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LAB 6: Orthonormal Bases, Orthogonal Projections, and QR decomposition

In this lab you will use MATLAB to study the following topics:

- Geometric aspects of vectors —*norm*, *dot product*, and *orthogonal projection* onto a line.
- The *Gram-Schmidt Algorithm* to change an independent set of vectors into an *orthonormal* set, and the associated $A = QR$ matrix factorization.
- The *orthogonal projection* of a vector onto a subspace.

Preliminaries

Reading from Textbook: In connection with this Lab, read through Sections 6.1 to 6.6 of the text and work the suggested problems for each section.

Tcodes and Script Files: For this lab you will need the Teaching Codes

```
grams.m,  linefit.m,  lsq.m, partic.m
```

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250 web page to your memory space (see Lab 3 for more details).

You will also need the m-files `rmat.m` and `rvect.m` from Lab 2. These files should already be in your directory. If you didn't do Lab 2, get a copy of that lab assignment and create these files as indicated there.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Be sure to answer all the questions in the lab assignment. Be sure your write-up begins with the following comment lines, filling in your information where appropriate.

```
% [Name]
% [Last 4 digits of RUID]
% Section [Section number]
% Math 250 MATLAB Lab Assignment #6
```

Enter `format compact`.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

7 Question 1. Norm, Dot Product, and Orthogonal Projection onto a Line

Generate random vectors $\mathbf{u}, \mathbf{v} \in \mathcal{R}^2$ by `u = rvect(2), v = rvect(2)`. Calculate `rank([u, v])` to determine whether they are linearly independent (this is also evident by inspection, since the vectors have integer entries). If the answer is not 2, then generate a new random pair of vectors and calculate the rank. Repeat until the rank is 2, and keep the vectors you generate in your lab writeup. Now use these vectors in the following calculations.

(2) **(a)** State (in words and symbols) the *triangle inequality* relating the norms $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}+\mathbf{v}\|$ for a general pair of vectors $\mathbf{u}$, $\mathbf{v}$. Then use MATLAB to show that your particular vectors $\mathbf{u}$, $\mathbf{v}$ satisfy this inequality. Note that $\|\,\mathbf{u}\,\|$ is calculated by the MATLAB command `norm(u)`.

(2) **(b)** State (in words and symbols) the *Cauchy-Schwarz inequality* relating the *dot product* $\mathbf{u}\cdot\mathbf{v}$ and the norms $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ for a general pair of vectors $\mathbf{u}$, $\mathbf{v}$. Then use MATLAB to show that your particular vectors $\mathbf{u}$, $\mathbf{v}$ satisfy this inequality. Note that dot product is calculated in MATLAB by `u'*v` when $\mathbf{u}$ and $\mathbf{v}$ are column vectors of the same size. The absolute value $|t|$ of a number t is calculated in MATLAB by `abs(t)`.

(1) **(c)** The *orthogonal projection* of the vector $\mathbf{u}$ onto the line $\mathcal{L}$ (one-dimensional subspace) spanned by the vector $\mathbf{v}$ is

$$\mathbf{w} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}$$

(see Figure 6.3 on page 366 of the text). Use MATLAB to calculate $\mathbf{w}$ for your vectors. Two vectors are *orthogonal* if their dot product is zero. Verify by MATLAB that the vector $\mathbf{z} = \mathbf{u} - \mathbf{w}$ is orthogonal to $\mathbf{v}$. (If the dot product is not exactly zero but is a very small number of size 10^{-13} for example, then the vectors are considered orthogonal for numerical purposes.)

(d) The formula for $\mathbf{w}$ in **(c)** can also be written as a matrix-vector product. Use MATLAB to obtain the matrix `P = v*inv(v'*v)*v'` (note carefully the punctuation and the order of the factors in this formula).

(2) Explain why P is a 2×2 matrix. Calculate by MATLAB that $P\mathbf{u}$ is the vector $\mathbf{w}$ for your $\mathbf{u}$ and $\mathbf{v}$.

13 Question 2. Gram-Schmidt Orthogonalization and QR Factorization

Generate three random vectors in $\mathcal{R}^3$ by

```
u1 = rvect(3), u2 = rvect(3), u3 = rvect(3)
```

Check whether they are linearly independent by calculating `rank([u1, u2, u3])`. If the answer is not 3, then generate a new random set of vectors and calculate the rank. Repeat until the rank is 3, and again, keep all the vectors you generate in your lab writeup. Now use these vectors in the following calculations.

(a) Since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are chosen at random, it is very unlikely that they are mutually orthogonal. To see this graphically using MATLAB, generate a line plotting parameter r and open a graphics window by the commands

```
r = 0:0.05:1; hold on
```

(be careful with the punctuation). Plot the three vectors in the graphics window as red, green, and blue dotted lines by the commands

```
plot3(r*u1(1),r*u1(2),r*u1(3), 'r:')
plot3(r*u2(1),r*u2(2),r*u2(3), 'g:')
plot3(r*u3(1),r*u3(2),r*u3(3), 'b:')
```

(use the up-arrow key ↑ to save typing).

(1) Using the Rotate 3D command, determine visually whether the vectors are mutually orthogonal or not. Insert a comment into your diary file.

(b) Now use the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to obtain an orthogonal basis for $\mathcal{R}^3$, following the Gram-Schmidt algorithm (see the proof of Theorem 6.6 on page 378 of the text). Set `v1 = u1`. Define the projection P_1 onto the span of $\mathbf{v}_1$ as in Question 1(d), and obtain $\mathbf{v}_2$ by removing the component of $\mathbf{u}_2$ in the direction $\mathbf{v}_1$:

```
P1 = v1*inv(v1'*v1)*v1', v2 = u2 - P1*u2
```

(1) Calculate the dot product to check that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are mutually orthogonal (within a negligible numerical error). Also add $\mathbf{v}_2$ to your graphics window as a dashed-dotted green line by

```
plot3(r*v2(1),r*v2(2),r*v2(3), 'g-.')
```

Using the Rotate 3D command, rotate the frame to see that the red line for $\mathbf{v}_1$ and the green line for $\mathbf{v}_2$ are orthogonal.
Now define P_2 as the projection onto the span of $\mathbf{v}_2$ and obtain $\mathbf{v}_3$ by removing the components of $\mathbf{u}_3$ in the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$:

```
P2 = v2*inv(v2'*v2)*v2', v3 = u3 - P1*u3 - P2*u3
```

(2) Calculate dot products by MATLAB to check that $\mathbf{v}_3$ is orthogonal to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (within a negligible numerical error). Add $\mathbf{v}_3$ to your plot as a dashed-dotted blue line by

```
plot3(r*v3(1),r*v3(2),r*v3(3), 'b-.')
```

Using the Rotate 3D command, rotate the frame to see that the red line for $\mathbf{v}_1$, the dash-dot green line for $\mathbf{v}_2$, and the dash-dot blue line for $\mathbf{v}_3$ are mutually orthogonal.

(2) Obtain a good alignment of the graph that shows orthogonality in perspective and then print the graph. Include the graph in your lab report.

(c) The last step in the Gram-Schmidt algorithm is to rescale the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to obtain an *orthonormal basis* for $\mathcal{R}^3$:

```
w1 = v1/norm(v1), w2 = v2/norm(v2), w3 = v3/norm(v3)
```

Define the matrix `Q = [w1, w2, w3]` and give written answers to the following questions.

(1) *(i)* Type, *as comments*, a symbolic (not numerical) hand calculation of the entries in the 3×3 matrix $Q^T Q$ in terms of the dot products $\mathbf{w}_i \cdot \mathbf{w}_j$. Use this to describe the orthonormal property of $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ in terms of $Q^T Q$.

(1) *(ii)* What is the *inverse matrix* Q^{-1}?

(2) Now check your answers to questions *(i)* and *(ii)* with MATLAB calculations.

(d) The Gram-Schmidt algorithm gives the QR factorization of a matrix (see the boxed statement on page 382 of the text). To illustrate this using MATLAB, set

```
A = [u1, u2, u3], R = Q'*A
```

(2) Verify by MATLAB that $A = Q * R$. Then give a symbolic (not numerical) proof of the fact that R is upper triangular, as follows.

(1) *(iii)* Let R have entries r_{ij}. Use the property $w_2 \cdot u_1 = 0$ to show that $r_{21} = 0$. Likewise, use the property $w_3 \cdot u_1 = w_3 \cdot u_2 = 0$ to show that $r_{31} = r_{32} = 0$.

5 Question 3. Orthogonal Projection onto a Subspace

Generate three random vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathcal{R}^5$ and the matrix A with these vectors as columns:

```
a1 = rvect(5); a2 = rvect(5); a3 = rvect(5); A = [a1, a2, a3]
```

Check whether they are linearly independent by calculating `rank(A)`. If the answer is not 3, then generate a new random set of vectors and calculate the rank. Repeat until the rank is 3, again keeping all the vectors you generate in your lab writeup. Now use these vectors and matrix in the following.

(a) Let $W = \mathrm{Col}(A)$ be the subspace of $\mathcal{R}^5$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. The teaching code `grams.m` carries out the steps of the Gram-Schmidt algorithm, just as you did step-by-step in Question 2. Calculate

```
Q = grams(A); w1 = Q(:,1), w2 = Q(:,2), w3 = Q(:,3)
```

by MATLAB.

(1) Calculate `Q'*Q` by MATLAB and explain why your answer shows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal set of vectors. (HINT: As in Question 2 (c)(i), relate the dot products to the entries of the matrix.)

(b) Orthogonal Decomposition $\mathbf{v} = \mathbf{w} + \mathbf{z}$: The orthogonal projection P from $\mathbf{R}^5$ onto the subspace W is given by the 5×5 matrix

```
P = w1*w1' + w2*w2' + w3*w3'
```

If $\mathbf{v} \in \mathcal{R}^5$ then $P\mathbf{v} = (\mathbf{w}_1 \cdot \mathbf{v})\mathbf{w}_1 + (\mathbf{w}_2 \cdot \mathbf{v})\mathbf{w}_2 + (\mathbf{w}_3 \cdot \mathbf{v})\mathbf{w}_3$ (see the boxed formula on page 376 of the text and Theorem 6.7 on page 392).

(1) Use MATLAB to calculate `P`. Then generate a random vector `v = rvect(5)` and calculate

$$\mathtt{w} = \mathtt{P} * \mathtt{v}, \quad \mathtt{z} = \mathtt{v} - \mathtt{w}.$$

If `z = 0` (this is very unlikely) generate another random vector until you get one with `z` not zero.

(1) Verify by MATLAB that `P*w = w` and `P*z = 0`. This shows that $\mathbf{w}$ is in the subspace W and that `z` is the component of $\mathbf{v}$ perpendicular W (see Figure 6.9 on page 388 of the text).

(2) **(c)** The projection matrix P onto the subspace W can be calculated directly from the matrix A, without first orthogonalizing the columns of A. Use MATLAB to obtain the matrix

```
PW = A*inv(A'*A)*A'
```

(see Theorem 6.8 on page 395 of the text). Check by MATLAB that `norm(PW - P)` is zero (up to negligible numerical error).

Final editing of lab write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Be sure to consult the instructions at the end of the MATLAB Demo assignment. Here is a summary:

Correct all typing errors and remove any unnecessary blank lines in your diary file. Your write-up must contain only the input commands that you typed which were required by the assignment (including `format compact` at the beginning), the output results generated by MATLAB, immediately following the corresponding input commands, your answers to the questions in the indicated places, and the indicated comments such as question numbers.

In particular, remove the commands `load, save, clear, format, help, diary`, **with the exception of** `format compact`, and remove any output from the commands `load, save, clear, format, help, diary`, as well.

Save the file as a plain text file.

Lab write-up submission guidelines: Preview the document before uploading and remove unnecessary page breaks and blank space. Make sure any images that need to be uploaded are in .jpeg or .pdf formats. Sakai will not allow you to upload files other than .pdf, .jpeg, or ,txt. Give yourself sufficient time to go through the submission procedure. Make allowances for computer and internet issues, as well as clock differences. *Late submissions will not be accepted.* Please be aware that both upload and submit steps need to be completed. If you do not complete both steps, your files will not be visible to the graders and you will receive a zero for the assignment.

Important: The submission of an unedited diary file without comments will be penalized by the removal of a significant number of points from the score.