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LAB 4: General Solution to $A\mathbf{x} = \mathbf{b}$

In this lab you will use MATLAB to study the following topics:

- The *column space* Col(A) of a matrix A
- The *null space* Null(A) of a matrix A.
- Particular solutions to an inhomogeneous linear equation $A\mathbf{x} = \mathbf{b}$.
- The complete solution of the equation $A\mathbf{x} = \mathbf{b}$.
- Application of the theory of inhomogeneous linear equations to a traffic flow problem.

Preliminaries

Reading from Textbook: The linear algebra ideas in this lab are found in Sections 4.1, 4.2, and 4.3 of the text. You should read the text and work the suggested problems for each section before working on this lab. Review the material in Section 1.4 of the text on *rank* and *nullity* of a matrix (pages 47-50). Many students find this lab harder than the others because it stresses the theory of linear systems of equations. This theory is at the heart of linear algebra, and it is worth extra effort to master these ideas.

Tcodes: For this lab you will need the Teaching Codes

```
nulbasis.m, elim.m, partic.m
```

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250C web page to your directory (see Lab 3 for more details).

Script files: You will need the MATLAB script files `rvect.m` and `rmat.m` from Lab 2 (if you didn't do Lab 2, get a copy of that assignment and follow the directions there to create these m-files). Be sure that you have set the path in MATLAB so that MATLAB can find your own m-files and the Teaching Codes.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Insert comments in your diary file as you work through the assignment. Be sure to answer all the questions in the lab assignment. Be sure your write-up begins with the following comment lines, filling in your information where appropriate.

```
% [Name]
% [Last 4 digits of RUID]
% Section [Section number]
% Math 250 MATLAB Lab Assignment #4
```

As usual, enter `format compact`.

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

$\boxed{6}$ Question 1. Visualizing the Column Space

In this question you will determine visually whether given vectors lie in the column space of a matrix.

Random Seed: Initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

Now generate a random 3×2 integer matrix by the MATLAB command `A = rmat(3,2)` and calculate `rank(A)`. Since A is a random matrix, the rank is very likely to be 2. If the rank is not 2, generate another A. Repeat the test until you get a matrix with rank 2. Use this matrix in the rest of the question.

If you need to generate more than one matrix, include all the matrices you generate in your lab report.

(1) **(a)** Define `u = A(:,1), v = A(:,2)` to be the column vectors for A. To graph the column space Col(A) of A, enter the MATLAB commands

```
 [s,t] = meshgrid((-1:0.1:1), (-1:0.1:1));
X = s*u(1)+t*v(1); Y = s*u(2)+t*v(2); Z = s*u(3)+t*v(3);
surf(X,Y,Z); axis square; colormap hot, hold on
```

A graph should appear in a separate window showing Col(A). From the `Tools` menu choose the command `Rotate 3D`. Using the mouse, position the cursor over the graph. Press and hold the left mouse button until a box appears to enclose the graph. Then move the mouse to rotate the graph in three dimensions.

(1) **(b)** Generate a random vector in $\mathbf{R}^3$ using the MATLAB m-file

```
b = rvect(3)
```

To graph the line Span($\mathbf{b}$) in the same figure as Col(A), enter the commands

```
r = -1:0.05:1;
plot3(r*b(1),r*b(2),r*b(3), '--')
```

Now determine whether $\mathbf{b}$ lies inside Col(A) graphically, using the `Rotate 3D` command. By rotating it enough, you should be able to see whether the *entire line* Span($\mathbf{b}$) lies in Col(A) or not.

Important: For *every* vector $\mathbf{v}$ the line Span($\mathbf{v}$) will intersect Col(A) in the point 0, since every subspace contains 0. You must look to see if *all* of the line through your vector $\mathbf{b}$ is in Col(A).
(Hint: Try to make Col(A) look like a line by viewing it edge-on.)

(1) Save the graph with a good choice of rotation showing whether or not $\mathbf{b}$ is in Col(A), as a .pdf or .jpeg file. This file will be uploaded to Sakai along with your lab write-up.

(1) **(c)** Can you find a vector $\mathbf{x} \in \mathbf{R}^2$ such that $A\mathbf{x} = \mathbf{b}$, where A is the matrix and $\mathbf{b}$ is the vector that you have generated? Explain why or why not using the graph from part **(b)**.

(1) **(d)** Generate a random vector lying in Col(A) using the commands

```
z = rand(2,1), c = A*z
```

Plot a new graph of Span($\mathbf{c}$) and Col(A) using

```
figure, surf(X,Y,Z); axis square; colormap hot, hold on
plot3(r*c(1),r*c(2),r*c(3), '+')
```

(1) Use `Rotate 3D` as in **(b)** to show that the entire line Span($\mathbf{c}$) is contained in Col(A). After making a good choice of rotation save the graph as a .pdf or .jpeg file. This file will be uploaded to Sakai along with your lab write-up.

$\boxed{6}$ Question 2. Reduced Row Echelon Form and Null Space

Generate a *partly random* 3×5 matrix A and its reduced row echelon form R. First generate a random 3×3 integer matrix and check its rank:

```
B = rmat(3,3), rank(B)
```

Since B is random, it is very likely to have rank 3. If not, generate another B until this is true. Keep all the matrices you generate in your lab report. Now use B to define a 3×5 matrix A and its reduced row echelon form R by

```
A = [B(:,1), B(:,2), 2*B(:,1) + 3*B(:,2), 4*B(:,1) - 5*B(:,2), B(:,3)],
R = rref(A)
```

(a) Use the definition of A in terms of B and the Column Correspondence Property (p. 129 of the text) to answer the following.

(1) Explain why columns #1, #2, and #5 are the pivot columns of A and R.

(1) Explain why column #3 of R is the vector $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and column #4 of R is the vector $\begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}$.

(b) Let V be the set of solutions to the homogeneous system of equations $A\mathbf{x} = 0$ (the *null space* of A).

(1) In the equation $A\mathbf{x} = 0$ (where $\mathbf{x} \in \mathbf{R}^5$), what are the *free variables* and what is $\dim V$?

(c) Use the MATLAB Teaching Code `nulbasis.m` to calculate the *special solutions* to the system of equations $A\mathbf{x} = 0$:

```
N = nulbasis(A)
```

The columns of N are the solutions to $A\mathbf{x} = 0$ obtained by setting one free variable to 1 and all the other free variables to 0. (see Example 8 on page 235 of the text). Define

```
v1 = N(:,1),   v2 = N(:,2)
```

(Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are 5-component *vectors*, not scalars.)

(1) Which component of $\mathbf{v}_1$ is 1 and which components of $\mathbf{v}_1$ are zero?

(1) Which component of $\mathbf{v}_2$ is 1 and which components of $\mathbf{v}_2$ are zero?

Check by MATLAB that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in Null(A).

(d) Now generate a random linear combination $\mathbf{x}$ of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by

```
s = rand(1), t = rand(1), x = s*v1 + t*v2
```

(Note that each occurrence of `rand(1)` generates a different random coefficient).

(1) Explain (without MATLAB) why $\mathbf{x}$ satisfies $A\mathbf{x} = 0$. Also explain why $\mathbf{x}$ satisfies $R\mathbf{x} = 0$. Then confirm this by MATLAB.

$\boxed{5}$ Question 3. Particular Solution to $A\mathbf{x} = \mathbf{b}$

Let A be a matrix of size $m \times n$ with $m \neq n$. The linear system $A\mathbf{x} = \mathbf{b}$ is called *underdetermined* if $m < n$ (more variables than equations). In this case there are more columns than pivots, so there are always free variables. Hence a solution $\mathbf{x}$ is never unique (the solution might or might not exist, depending on the choice of $\mathbf{b}$). The system is called *overdetermined* if $m > n$ (more equations than variables). In this case, there are more rows than pivots, and hence there are always vectors $\mathbf{b} \neq 0$ for which there is no solution $\mathbf{x}$. In both cases the matrix A is *not* square, so the system can never be solved by finding an inverse matrix for A.

(a) Particular Solution (overdetermined system): Generate a random 5×3 integer matrix A (the coefficient matrix for an *overdetermined* system of 5 equations in 3 unknowns) and its reduced row echelon form R by

```
A = rmat(5, 3),   R = rref(A)
```

Since A is random, the matrix $A(:, 1:3)$ is very likely to have rank 3. If the rank of $A(:, 1:3)$ is not 3, generate a new matrix A until the rank of $A(:, 1:3)$ is 3, and use this matrix. Keep all the matrices you generate in your lab report.

(1) Explain (without MATLAB) why there exist vectors $\mathbf{b} \in \mathbf{R}^5$ such that equation $A\mathbf{x} = \mathbf{b}$ does not have a solution. (See Theorem 1.6 on page 70 of the text).

The following MATLAB command will generate a random 5×1 vector $\mathbf{b}$ and try to find a particular solution to $A\mathbf{x} = \mathbf{b}$:

```
b = rmat(5,1),   xp = partic(A, b)
```

The answer should be `xp = []` (empty vector, meaning no solution).

(1) Now use the (partly) random vector

```
b = rand(1)*A(:,1) + rand(1)*A(:,2) + rand(1)*A(:,3)
```

and calculate `xp = partic(A, b)`. Then use MATLAB to check that `A*xp = b`.

(1) Explain (without MATLAB) why the special form of this $\mathbf{b}$ guarantees that there is a solution to $A\mathbf{x} = \mathbf{b}$. (See Theorem 1.5 on page 50 of the text).

(b) Particular Solution (underdetermined system): Generate a random 3×5 integer matrix A (the coefficient matrix for an *underdetermined* system of 3 equations in 5 unknowns) and its reduced row echelon form R by

```
A = rmat(3, 5),   R = rref(A)
```

Since A is random, the matrix $A(:, 1:3)$ is very likely to have rank 3. If the rank of $A(:, 1:3)$ is not 3, generate a new matrix A until the rank of $A(:, 1:3)$ is 3, and use this matrix. As usual, keep all the matrices you generate in your lab report.

(1) Explain (without MATLAB) why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for *every* vector $\mathbf{b} \in \mathbf{R}^3$. (See Theorem 1.6 on page 70 of the text).

Now generate a random 3×1 vector $\mathbf{b}$ and use the Teaching Code `partic.m` to find a particular solution to $A\mathbf{x} = \mathbf{b}$ by

```
b = rmat(3,1),   xp = partic(A, b)
```

This is the solution with all the free variables set to zero (see pages 30-31 of the text).

(1) Why are the entries in row 4 and 5 of `xp` zero?

Check by MATLAB that `A*xp = b`.

$\boxed{2}$ Question 4. General Solution to $A\mathbf{x} = \mathbf{b}$

Let A be the 3×5 matrix and $\mathbf{b}$ the 3×1 vector that you generated in Question #3(b). The *general solution* to an inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ is obtained by adding a vector from the *null space* of A to the particular solution `xp`.

(1) **(a)** Use the T-code `nulbasis.m` to obtain the 5×2 matrix

```
N = nulbasis(A)
```

Set `v1 = N(:,1), v2 = N(:,2)` and form a random *general solution*

```
x = xp + rand(1)*v1 + rand(1)*v2
```

to $A\mathbf{x} = \mathbf{b}$. Check by MATLAB that `A*x` is the vector `b`.

(1) **(b)** Now solve the equation $A\mathbf{x} = \mathbf{b}$ with the extra condition that $\mathbf{x}$ should be of the form

$$\mathbf{x} = [x_1, x_2, x_3, -9, 8]^{\mathrm{T}}$$

For this, you must choose particular scalars `c1` and `c2` in the general solution `x = xp + c1*v1 + c2*v2`. (HINT: Look at the free variables.) Then check by MATLAB that `A*x` is the vector `b`.

6 Question 5. Analysis of Traffic Flow

This question is based on the Traffic Flow Example in Section 2.2 of the text (pages 110-111). Read through that example before working the problem. Use the matrices A, B, C and $M = CBA$ from this example in the following. *Be careful when you copy the entries in these matrices from the text into* MATLAB.

(a) Generate the following:

```
x = 1000*rvect(2), y = A*x, z = B*y, w = C*z
```

by MATLAB. The vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ describe the random traffic flows through succesive parts of the network.

(2) Calculate `[1 1]*x, [1 1 1]*y, [1 1 1]*z, [1 1 1 1]*w` using MATLAB.

(1) Explain the meaning of each answer in terms of the traffic flow through a particular part in the network.

(1) **(b)** Suppose that on a particular day the traffic flow gives the vector `y = [270 126 704]'`. Give the equation relating `y` and `x` and solve it for `x` using MATLAB as in Question 3. Note that these vectors `x` and `y` are not the same as the random vectors you generated in part **(a)**. Now you are starting with `y` and trying to solve for `x`.

(1) Is the entering traffic flow vector `x` that you just found uniquely determined in this case? Explain using the general theory of solving $A\mathbf{x} = \mathbf{b}$ from Question 4.

(1) **(c)** Let `w = [100 200 300 400]'` . Give the equation relating `w` and `x` and use MATLAB as in Question 3 to determine if `w` can be an output traffic vector. Note that these vectors `x` and `w` are not the same as the random vectors you generated in part **(a)**. Now you are starting with `w` and trying to find `x`. (HINT: Calculate `rref([M w])` to see if the equation relating `x` and `w` is consistent.)

Final editing of lab write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Be sure to consult the instructions at the end of the MATLAB Demo assignment. Here is a summary:
Correct all typing errors and remove any unnecessary blank lines in your diary file. Your write-up must contain only the input commands that you typed which were required by the assignment (including `format compact` at the beginning), the output results generated by MATLAB, immediately following the corresponding input commands, your answers to the questions in the indicated places, and the indicated comments such as question numbers.
In particular, remove the commands `load, save, clear, format, help, diary`, **with the exception of** `format compact`, and remove any output from the commands `load, save, clear, format, help, diary`, as well.
Save the file as a plain text file.

Lab write-up submission guidelines: Preview the document before uploading and remove unnecessary page breaks and blank space. Make sure any images that need to be uploaded are in .jpeg or .pdf formats. Sakai will not allow you to upload files other than .pdf, .jpeg, or ,txt. Give yourself sufficient time to go through the submission procedure. Make allowances for computer and internet issues, as well as clock differences. *Late submissions will not be accepted.* Please be aware that both upload and submit steps need to be completed. If you do not complete both steps, your files will not be visible to the graders and you will receive a zero for the assignment.

Important: The submission of an unedited diary file without comments will be penalized by the removal of a significant number of points from the score.