Revised 1/10/07

LAB 4: General Solution to $A\mathbf{x} = \mathbf{b}$

In this lab you will use MATLAB to study the following topics:

- The *column space* Col(A) of a matrix A
- The *null space* Null(A) of a matrix A.
- Particular solutions to an inhomogeneous linear equation $A\mathbf{x} = \mathbf{b}$.
- The complete solution of the equation $A\mathbf{x} = \mathbf{b}$.
- Application of the theory of inhomogeneous linear equations to a traffic flow problem.

Preliminaries

Reading from Textbook: The linear algebra ideas in this lab are found in Sections 4.1, 4.2, and 4.3 of the text. You should read the text and work the suggested problems for each section before working on this lab. Review the material in Section 1.4 of the text on *rank* and *nullity* of a matrix (pages 42-44).

Tcodes: For this lab you will need the Teaching Codes

```
nulbasis.m, elim.m, partic.m
```

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250C web page to your directory (see Lab 3 for more details).

Script files: You will need the MATLAB script files `rvect.m` and `rmat.m` from Lab 2 (if you didn't do Lab 2, get a copy of that assignment and follow the directions there to create these m-files). Be sure that you have set the path in MATLAB so that MATLAB can find your own m-files and the Teaching Codes.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Insert comments in your diary file as you work through the assignment. Be sure to answer all the questions in the lab assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Visualizing the Column Space

In this question you will determine visually whether given vectors lie in the column space of a matrix.

(a) Generate a random 3×2 matrix by the MATLAB command

```
A = rmat(3,2)
```

Define $\mathbf{u}$ and $\mathbf{v}$ to be the column vectors for A:

```
u= A(:,1), v = A(:,2)
```

To graph the column space Col(A) of A, enter the MATLAB commands

```
 [s,t] = meshgrid((-1:0.1:1), (-1:0.1:1));
X = s*u(1)+t*v(1); Y = s*u(2)+t*v(2); Z = s*u(3)+t*v(3);
surf(X,Y,Z); axis square; colormap hot, hold on
```

A graph should appear in a separate window showing Col(A). From the `Tools` menu choose the command `Rotate 3D`. Using the mouse, position the cursor over the graph. Press and hold the left mouse button until a box appears to enclose the graph. Then move the mouse to rotate the graph in three dimensions.

(b) Generate a random 3-vector using the MATLAB m-file

```
b = rvect(3)
```

To graph the line Span($\mathbf{b}$) in the same figure as Col(A), enter the commands

```
r = -1:0.05:1;
plot3(r*b(1),r*b(2),r*b(3), '+')
```

Now determine whether $\mathbf{b}$ lies inside Col(A) graphically, using the `Rotate 3D` command. By rotating it enough, you should be able to see whether the entire line Span($\mathbf{b}$) lies in Col(A) or not. (Hint: try to make Col(A) look like a line.) Print a copy of the graph with a good choice of rotation and include it with your write-up.

(c) Generate a random vector lying in Col(A) using the commands

```
z = rand(2,1), c = A*z
```

Plot a new graph of Span($\mathbf{c}$) and Col(A) using

```
figure, surf(X,Y,Z); axis square; colormap hot, hold on
plot3(r*c(1),r*c(2),r*c(3), '+')
```

Using the same procedure as before, use `Rotate 3D` to show that the entire line Span($\mathbf{c}$) is contained in Col(A). Print the graph with good choice of rotation and include it with your write-up.

(d) Is the equation $A\mathbf{x} = \mathbf{b}$ solvable, for these values of A and $\mathbf{b}$? Explain why or why not using the graph from part **(b)**.

(e) Is the equation $A\mathbf{x} = \mathbf{c}$ solvable? Explain why or why not using the graph from part **(c)**.

Question 2. Null Space

Generate a *partly random* 3×6 matrix A and its reduced row echelon form R by

```
B = rmat(3, 2); C = rmat(3,2); A = [B, 3*B, C], R = rref(A)
```

Let V be the subspace of $\mathcal{R}^6$ given by the homogeneous system of equations $A\mathbf{x} = 0$ (the *null space* of A). Now answer the following questions:

(a) In the system of linear equations $A\mathbf{x} = 0$ (where $\mathbf{x} \in \mathcal{R}^6$), what are the *free variables*? What is $\dim V$?

(b) Use the MATLAB Teaching Code `nulbasis.m` to calculate the *special solutions* to the system of equations $A\mathbf{x} = 0$:

```
N = nulbasis(A)
```

The columns of N are the solutions to $A\mathbf{x} = 0$ obtained by setting one free variable to 1 and all the other free variables to 0 (see Example 8 on page 205). Define

```
v1 = N(:,1),   v2 = N(:,2),   v3 = N(:,3)
```

(Notice that each $\mathbf{v}_i$ is a 6-component *vector*, not a scalar.)

(i) Which component of $\mathbf{v}_1$ *must be* 1?

(ii) Which components of $\mathbf{v}_1$ *must be* 0?

Answer questions *(i)* and *(ii)* also for $\mathbf{v}_2$ and $\mathbf{v}_3$. Check by MATLAB that these three vectors are all in Null(A).

(c) Now generate a random linear combination $\mathbf{x}$ of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ by

```
x = rand(1)*v1 + rand(1)*v2 + rand(1)*v3
```

(Each occurrence of `rand(1)` generates a different random coefficient). Check by MATLAB that $A\mathbf{x} = 0$.

(i) Explain (without MATLAB) why $\mathbf{x}$ also satisfies $R\mathbf{x} = 0$.

Confirm by MATLAB that $R\mathbf{x} = 0$.

Question 3. Particular Solutions to $A\mathbf{x} = \mathbf{b}$

Let A be a matrix of size $m \times n$. The linear system $A\mathbf{x} = \mathbf{b}$ is called *underdetermined* if $m < n$ (more variables than equations). It is *overdetermined* if $m > n$ (more equations than variables). In both cases the matrix A is *not* square, so the system can never be solved by finding an inverse matrix for A.

(a) Particular Solution (underdetermined system): Generate a random 3×5 matrix A (the coefficient matrix for an *underdetermined* system of 3 equations in 5 unknowns) and its reduced row echelon form R by

```
A = rmat(3,5),   R = rref(A)
```

Now generate a random 3×1 vector $\mathbf{b}$ and use the Teaching Code `partic.m` to find a particular solution to $A\mathbf{x} = \mathbf{b}$ by

```
b = rmat(3,1),   x = partic(A, b)
```

This is the solution with all the free variables set to zero (see pages 30-31 of the text). Check with MATLAB that $A\mathbf{x} = \mathbf{b}$. Repeat for another random vector $\mathbf{b}$, using the same matrix A.

(i) What entries in $\mathbf{x}$ are zero both times?

(ii) Which columns of R correspond to free variables?

Calculate `rank(A)` and `rank([A, b])` using MATLAB.

(iii) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for *every* vector $\mathbf{b}$ in this case? Why?

(See Theorem 1.4 on page 44 of the text).

(b) Particular Solution (overdetermined system): Generate a random 5×3 matrix `A = rmat(5,3)` (the coefficient matrix for an *overdetermined* system of 5 equations in 3 variables). The following MATLAB command will generate a random 5×1 vector $\mathbf{b}$ and try to find a particular solution to $A\mathbf{x} = \mathbf{b}$:

```
b = rmat(5,1),   x = partic(A, b)
```

Calculate `rank(A)` and `rank([A, b])` using MATLAB.

(i) Why is there no solution to $A\mathbf{x} = \mathbf{b}$ for a completely random choice of $\mathbf{b}$? (See Theorem 1.4 on page 44 of the text).

Now use the (partly) random vector

```
b = rand(1)*A(:,1) + rand(1)*A(:,2) + rand(1)*A(:,3)
```

and calculate `x = partic(A, b)`. Write comments to answer the following:

(ii) Explain why the special form of the vector $\mathbf{b}$ guarantees that there always is a solution $\mathbf{x}$ for any choice of the random coefficients.

(iii) Since there is a solution to $A\mathbf{x} = \mathbf{b}$ for this $\mathbf{b}$, what is the rank of the augmented matrix `[A, b]`? (See Theorem 1.4 on page 44 of the text.)

Check your answer to *(iii)* using MATLAB.

Question 4. General Solution to $A\mathbf{x} = \mathbf{b}$

The *general solution* to an inhomogeneous linear system is obtained by adding a vector from the *null space* of A to a particular solution, as follows.

(a) Execute the commands

```
A = rmat(3,5), b = rmat(3,1), N = nulbasis(A), xp = partic(A,b)
```

Set `v1 = N(:,1), v2 = N(:,2)` and form a random *general solution*

```
x = xp + rand(1)*v1 + rand(1)*v2
```

to $A\mathbf{x} = \mathbf{b}$. Check by MATLAB that $A\mathbf{x} - \mathbf{b} = 0$.

(b) Now solve the equation $A\mathbf{x} = \mathbf{b}$ with the extra condition that $\mathbf{x}$ should be of the form

$$\mathbf{x} = [x_1,\, x_2,\, x_3,\, -9,\, 8]^{\mathrm{T}}$$

For this, you must choose particular scalars c_1 and c_2 so that $\mathbf{x} = \mathbf{x}_p + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. (Hint: Look at the free variables in $\mathbf{x}$, $\mathbf{x}_p$, $\mathbf{v}_1$, and $\mathbf{v}_2$). Then check your answer by calculating $A\mathbf{x} - \mathbf{b}$ with MATLAB.

Question 5. Analysis of Traffic Flow

This question is based on the Traffic Flow Example in Section 2.2 text (pages 101-103). Read through that example before working the problem. Use the matrices A, B, C and $M = CBA$ from this example in the following. The vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ describe the traffic flows through succesive parts of the network. Use the techniques for solving systems of linear equations from Questions 3 and 4.

(a) Generate

```
x = 1000*rvect(2), y = A*x, z = B*y, w = C*z
```

by MATLAB. Then calculate

```
[1 1]*x, [1 1 1]*y, [1 1 1]*z, [1 1 1 1]*w
```

Explain the meaning of the answers in terms of the traffic flow through the network.

(b) Suppose that on a particular day the vector `y = [270 126 704]'` . Use MATLAB to find the input traffic vector `x` on that day. Is `x` uniquely determined by these data? Explain using the general theory of solving $A\mathbf{x} = \mathbf{b}$.

(c) Let `w = [100 200 300 400]'` . Use MATLAB to determine if `w` can be the output traffic vector on a particular day. Explain using the general theory of solving $A\mathbf{x} = \mathbf{b}$.

(d) Is it possible that $w_3 = 350$ and $w_4 = 150$ on a particular day? Is this enough information to determine the input traffic vector `x`? Explain using the general theory of solving $A\mathbf{x} = \mathbf{b}$.

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Remove all errors and other material that is not directly related to the questions. Your write-up should only contain the required MATLAB calculations and the answers to the questions. Preview the document before printing and remove unnecessary page breaks and blank space.