Revised 1/10/07

LAB 3: *LU* Decomposition and Determinants

In this lab you will use MATLAB to study the following topics:

- The LU decomposition of an invertible square matrix A.
- How to use the LU decomposition to solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.
- Comparison of the computation time to solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination vs. solution by LU decomposition of A.
- The determinant of a square matrix, how it changes under row operations and matrix multiplication, and how it can be calculated efficiently by the LU decomposition.
- The geometric properties of special types of matrices (rotations, dilations, shears).

Preliminaries

Reading from Textbook: Before beginning the Lab, read through Sections 2.5, 3.1 and 3.2 of the text and work the suggested problems for these sections.

Tcodes: In this course you will use some instructional MATLAB *m-files* called *Tcodes*. To obtain any of these files, use a web browser and go to the Math Department Home page *http://www.math.rutgers.edu*. Click on *course materials*, then on *Math 250 Introduction to Linear Algebra* , and then on *MATLAB Teaching Codes*. You will see a directory of the m-files. Click on the particular m-files that you need. Then in the menu bar click on *Files* and *Save As*. Fill in the directory information that is requested.

For this lab you will need the Teaching Codes

```
cofactor.m,  splu.m  plot2d.m,  house.m
```

Before opening MATLAB to work on the Lab questions you should copy these codes to your directory by the method described above.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #3
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
       ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Social Security number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Row Operations and LU Factorization

In this problem you will use MATLAB to carry out elementary row operations and to obtain the matrix factorization $A = LU$ for a square 3×3 matrix A.

(a) Generate a random 3×3 matrix A using the m-file `rmat(m,n)` that you created in Lab 2:

```
A = rmat(3,3), U = A
```

Here U has the initial value A, but at the end of the LU algorithm U will be upper triangular.

(b) Use the MATLAB editor to create an m-file called `col1.m` with the following MATLAB commands:

```
L1 = eye(3);
L1(2,:) = L1(2,:) + (U(2,1)/U(1,1))*L1(1,:);
L1(3,:) = L1(3,:) + (U(3,1)/U(1,1))*L1(1,:);
L1
```

(notice the use of ; to suppress screen output of the intermediate results). Execute this file by typing `col1` at the MATLAB prompt. If you get a division by zero error message, go back to step (a) and generate another A and U.

(i) What row operations do you apply to the 3×3 identity matrix to obtain the matrix L_1?

Use MATLAB to calculate L_1^{-1} (if M is a square matrix that has an inverse, then the MATLAB command `inv(M)` will calculate the inverse matrix M^{-1}).

(ii) How are the matrix entries of L_1^{-1} obtained from those of L_1?

Now use MATLAB to replace the current value of the matrix U by the new value $L_1^{-1} * U$ (remember that the command $X = Y$ in MATLAB means to replace the current value of the variable X by the current value of the variable Y).

(iii) Describe how the new value of U is obtained from the old value of U by row operations.

(c) Use the MATLAB editor to create an m-file called `col2.m` with the commands

```
L2 = eye(3);
L2(3,:) = L2(3,:) + (U(3,2)/U(2,2))*L2(2,:);
L2
```

This will be used with the matrix U modified as in (b). Execute this file by typing col2 at the MATLAB prompt. If you get a division by zero error message, go back to step (a) and start again (this is very unlikely to happen since A is completely random). Use MATLAB to calculate L_2^{-1}.

(i) Describe how the matrix entries of L_2^{-1} are obtained from those of L_2.

Now use MATLAB to replace the current value of the matrix U by the new value $L_2^{-1} * U$.

(ii) Describe how the new value of U is obtained from the old value of U by row operations.

(d) Use MATLAB to get $L = L_1 * L_2$ and verify that $A = L * U$ (where U has the value from (c)).

(i) Describe how the entries of L are obtained from those of L_1 and L_2. (See the boxed statement on page 140 of the text.)

Question 2. Using LU Factorization to Solve $A\mathbf{x} = \mathbf{b}$

(a) Inverting A, L and U: Use MATLAB to calculate the inverses of the matrices L and U that you obtained in Question #1. Then write comments to answer the following questions.

> *(i)* Which entries in `inv(L)` and `inv(U)` are *always* zero (no matter what random matrix A you generate)?
>
> *(ii)* Which entries in `inv(L)` are *always* 1?
>
> *(iii)* For the matrices L_1 and L_2, you found in Question #1 that the inverse matrices are simply obtained by putting a minus sign in front of the entries below the main diagonal. Does this method give the correct inverse matrix for L? Justify your answer.

(b) Solving $A\mathbf{x} = \mathbf{b}$ using L and U (See Example 4 on page 140 of the text): Use the m-file `rvect.m` from Lab 2 to generate a random vector `b = rvect(3)`. Calculate the solution

```
c = inv(L)*b
```

to the triangular system $L\mathbf{c} = \mathbf{b}$. Then calculate the solution

```
x = inv(U)*c
```

to the triangular system $U\mathbf{x} = \mathbf{c}$. Finally, calculate $A\mathbf{x}$ and check that it is $\mathbf{b}$ (since the entries in $\mathbf{b}$ are integers, this should be obvious by inspection).

Question 3. LU versus rref for solving $A\mathbf{x} = \mathbf{b}$

In this question you will compare the speed of two methods of solving the equation $A\mathbf{x} = \mathbf{b}$ when A is an invertible square matrix. You will use the MATLAB `tic` and `toc` commands to measure the computation times.

> **Important:** Be sure to use the semicolon ; after each command as indicated below so that the matrices and vector in this question are *not* displayed or included in your diary file. Do not print or include these large matrices and vectors in your lab write-up.

Generate a random 100×100 matrix A, a vector $\mathbf{b} \in \mathcal{R}^{100}$, and calculate the LU decomposition of A by

```
A = rand(100) ;   b = rand(100,1); [L U] = lu(A);
```

(a) Solve $A\mathbf{x} = \mathbf{b}$ by using the reduced row echelon form (Gaussian elimination). The last column $\mathbf{y}$ of the augmented matrix $R = \mathrm{rref}([A\ \mathbf{b}])$ satisfies $A\mathbf{y} = \mathbf{b}$ because `rref(A)` is the identity matrix if A is a random square matrix.

```
tic; R = rref([A b]); y = R(:,101); toc
```

Define the number `rreftime` to be the `elapsed_time` given by the MATLAB output in this case.

(b) Next, solve $A\mathbf{x} = \mathbf{b}$ by using the LU decomposition of A:

```
tic; c = inv(L)*b; x = inv(U)*c; toc
```

Define the number `lutime` to be the `elapsed_time` given by the MATLAB output in this case.

(c) Check that the solutions from parts **(a)** and **(b)** are the same (up to round-off error) by calculating `norm(x - y)` (the norm function gives the length of the vector $\mathbf{x} - \mathbf{y}$).

(d) According to the table on page 143 of the text, the computation time for Gaussian elimination is approximately $2cn^3/3$, while the time for the LU method (after the L, U factors are already calculated) is approximately $2cn^2$. Here c is a constant depending on the processing speed of the arithmetic chip in the computer and n is the number of equations.

> *(i)* What is the predicted ratio `rreftime/lutime` when $n = 100$?
>
> *(ii)* Calculate the observed ratio `rreftime/lutime` from the calculations in parts **(a)** and **(b)**.

Comment: The basic arithmetic operations in computers are now so fast that a large proportion of the elapsed computing time consists of data transfer. Thus your answer to *(ii)* will probably not agree with *(i)*.

Question 4. The Determinant Function

(a) Cofactor Expansion: The Teaching Code m-file `cofactor.m` calculates the matrix of cofactors of a square matrix. Generate a random 4×4 integer matrix `a = rmat(4,4)`. Then use MATLAB to calculate the cofactor matrix `c = cofactor(a)`. Now use MATLAB to calculate the four sums

```
a(1,1)*c(1,1) + a(1,2)*c(1,2) + a(1,3)*c(1,3) + a(1,4)*c(1,4)
a(2,1)*c(2,1) + a(2,2)*c(2,2) + a(2,3)*c(2,3) + a(2,4)*c(2,4)
a(1,3)*c(1,3) + a(2,3)*c(2,3) + a(3,3)*c(3,3) + a(4,3)*c(4,3)
a(1,4)*c(1,4) + a(2,4)*c(2,4) + a(3,4)*c(3,4) + a(4,4)*c(4,4)
```

(use the up-arrow key ↑ and edit the line to save retyping).

> *(i)* Use Theorem 3.1 (page 178) and Theorem 3.4 (page 188) to explain the answers in this calculation.

Check by using MATLAB to calculate $\det(a)$.

(b) Triangular Matrices: Generate a random 5×5 matrix A and a random upper triangular matrix U by

```
A = rmat(5,5), U = triu(A)
```

Calculate the product `A(1,1)*A(2,2)*A(3,3)*A(4,4)*A(5,5)` of the diagonal entries of A.

> *(i)* Can you obtain $\det(A)$ from this single term? Explain.

Confirm your answer by a MATLAB calculation. Now calculate the the corresponding product `U(1,1)*U(2,2)*U(3,3)*U(4,4)*U(5,5)` for the matrix U.

> *(ii)* Can you obtain $\det(U)$ from this single term? Explain.

Confirm your answer by a MATLAB calculation.

(c) Row Operations: Generate a 5×5 random integer matrix `A = rmat(5,5)`. Then swap the first and second row of A to get the matrix B using the following commands:

```
B = A; B(2,:)  = A(1,:); B(1,:)  = A(2,:)
```

Use properties of the determinant function to answer the following:

> *(i)* What is the relation between $\det(A)$ and $\det(B)$?

Check your answer by calculating `det(A)` and `det(B)` using MATLAB. Next, let C be the matrix obtained from A by multiplying the first row of A by 10 and adding to the second row of A using the following commands:

```
C = A; C(2,:)  = A(2,:)  + 10*A(1,:)
```

Use properties of the determinant function to answer the following:

> *(ii)* What is the relation between $\det(A)$ and $\det(C)$?

Check your answer by MATLAB. Finally, let D be the matrix obtained from A by multiplying the first row of A by 10:

```
D = A; D(1,:)  = 10*A(1,:)
```

Use properties of the determinant function to answer the following:

(iii) What is the relation between $\det(A)$, $\det(D)$, and $\det(10 * A)$?

Check your answers by MATLAB.

(d) Multiplicative Property: Generate a random 5×5 integer matrix `A = rmat(5,5)`. Then modify A by setting `A(1,1)=0; A(2,1) = 0`. The reduction of the (modified) matrix A to row echelon form can be expressed in terms of a matrix factorization as $PA = LU$. Here P is a *permutation matrix* that expresses the row interchanges that are needed to apply Gaussian elimination to A, and L and U give the LU decomposition of PA (read pages 144-146 of the text).

(i) For the modified matrix A, explain why P will not be the identity matrix.

You can calculate the $PA = LU$ factorization by using the T-code `splu.m`:

```
[P, L, U, sign] = splu(A)
```

Here `sign` gives $\det(P)$, which is +1 for an even number of row interchanges to transform P into the identity matrix, and -1 for an odd number of row interchanges. Check (by MATLAB) that $PA = LU$. Then write comments to answer the following.

(ii) What is $\det(P)$? Why? Compare your answer with the value of `sign` that MATLAB has calculated.

(iii) What is $\det(L)$? Why?

(iv) How many scalar multiplications are needed to calculate $\det(U)$? Why?

(v) What is the relation between $\det(A)$ and $\det(U)$? Why?

Check your answer to *(v)* by MATLAB.

Question 5. Geometry and Matrices

This question uses MATLAB to illustrate the geometric meaning of some special types of matrices. At the MATLAB prompt type

```
H = house; plot2d(H), hold on
```

A graphics window should open and display a crude drawing of a house. The matrix `H` contains the coordinates of the endpoints of the line segments making up the drawing.

(a) Rotations: Generate a matrix Q by

```
t = pi/6; Q = [cos(t), -sin(t); sin(t), cos(t)]
```

Let Q act on the house by `plot2d(Q*H)` .

(i) How has the house been changed?

(ii) Calculate $\det(Q)$. What does this tell you about the area inside the transformed house? (see page 181 of the text).

Repeat this process with `t = - pi/3` (use ↑ to save typing) and answer *(i)* and *(ii)* in this case. Print the graph with the three house images on the same figure.

(b) Dilations: Clear the graphics window and generate a new plot of the house as above. Generate a matrix D by

```
r = .9; D = [r, 0; 0, 1/r ]
```

Let D act on the house by `plot2d(D*H)`.

(i) How has the house been changed?

(ii) Calculate $\det(D)$. What does this tell you about the area inside the transformed house?

Repeat this process with `r = .8` and answer *(i)* and *(ii)* in this case. Print the graph with the three house images on the same figure.

(c) Shearing Transformations: Clear the graphics window and generate a new plot of the house as above. Generate a matrix T by

```
t = 1/2; T = [1, t; 0, 1]
```

Now let T act on the house by `plot2d(T*H)`.

(i) How has the house been changed?

(ii) Calculate $\det(D)$. What does this tell you about the area inside the transformed house?

Repeat this process with `t = - 1/2` and answer *(i)* and *(ii)* in this case.

(iii) What is the relation between the transformations with $t = 1/2$ and $t = -1/2$?

Print the graph with the three house images on the same figure.

Final Editing of Lab Write-up:

After you have worked through all the parts of the lab assignment, edit your diary file. Include the MATLAB calculations, but remove errors that you made in entering commands and remove other material that is not directly related to the questions. *Be sure not to print out any of the 100 $\times$ 100 matrices from question 3.*