Revised 1/10/07

## LAB 2: Linear Equations and Matrix Algebra

In this lab you will use MATLAB to study the following topics:

- Solving a system of linear equations by using the reduced row echelon form of the augmented matrix of the system.
- Forming linear combinations of a set of vectors and the fundamental concepts of *linear dependence* and *linear independence*.
- Matrix multiplication and its properties.
- Application of Adjacency Matrices to Communication Networks.

### *Preliminaries*

**Reading from Textbook:** Before beginning the Lab, read through Sections 1.3, 1,4, 1.6, 1.7, and 2.1 of the text and work the suggested problems for these sections.

**MATLAB Help:** In Lab 1 you learned the basic MATLAB commands. Remember that every MATLAB command is documented in a `help` file, which you can access during a MATLAB session. For example typing `help format` gives information about the command `format`. Go to the mathematics department web site for this course for additional MATLAB documents if you want further information. For this lab, create a `diary` file and edit it as you did for Lab 1.

**Script Files:** For more complicated MATLAB calculations you should use *scripts.* A script contains one or several MATLAB commands and is stored as a text file with a descriptive name such as `mymatrix.m`, for example (the extension `*.m` is required). When you type the name of a script (without the extension `*.m`) at the MATLAB command prompt the commands within the script are executed, affecting the variables in the global workspace. Such scripts are called *m-files.* The advantage of having scripts is that you can execute the commands in the script at any time by typing the *name* of the script instead of the *contents* of the script.

**Writing Scripts:** When you need to write a script file for this lab and subsequent labs, use the following procedure: Start MATLAB and click on *File*, then *New.* Move the pointer to the right and click on *m-Files.* This will open the MATLAB Editor/Debugger Window, and you can type the script commands in this window. You can take any m-file, edit it (just as you would edit any text file), and then save it under a different name to obtain a new m-file.

**Running Scripts:** After you have created an m-file and saved it to your directory, you must set the *Path* so that MATLAB can find this file. Click on *File*, then *Set Path* and follow the directions to add your directory to the list of path names.

**Script Files for Lab 2:** Use the text editor in MATLAB to create the following MATLAB *function* m-files:

**(a)** **`rvect.m`:** Create a *function* m-file with the commands

```
function v = rvect(m)
v = fix(10*rand(m,1));
```

(note the semicolon on the end of the second line). Save this file under the name `rvect.m` (be sure that you have set the *Path* as described above so that MATLAB can find this m-file). Test the file by clicking on the MATLAB window and typing `v = rvect(3)` at the MATLAB prompt. You should get a column vector $\mathbf{v} \in \mathbf{R}^3$ with entries that are (random) integers between 0 and 9. Now type `u = rvect(3)`. You will get another random column vector $\mathbf{u} \in \mathbf{R}^3$. Type `v` at the prompt. You should get the *same* vector $\mathbf{v}$ as before.

Note that the name **v** in the `rvect` function file is a *local variable*; you can assign any name to the output. If you have already defined a vector **v** in your work space, it is not changed when you generate **u** by `rvect`.

**(b)** **`rmat.m`:** Create another function m-file with the commands

```
function A = rmat(m,n)
A = fix(10*rand(m,n));
```

(note the semicolon on the end of the second line). Save this file under the name `rmat.m`. Then test the file by clicking on the MATLAB window and typing `A = rmat(3, 5)` at the MATLAB prompt. You should get a $3 \times 5$ matrix $A$ with entries that are (random) integers between 0 and 9.

**Lab Write-up:** After writing and testing the script files, open your diary file (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #2
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
        ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

**Random Seed:** When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Social Security number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

**The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.**

## Question 1. Solving $A\mathbf{x} = \mathbf{b}$

In this question you will find the general solution $\mathbf{x} \in \mathbf{R}^3$ to a linear system $A\mathbf{x} = \mathbf{b}$ of 3 equations in 5 variables $x_1, x_2, x_3, x_4, x_5$. Here $A$ is the $3 \times 5$ coefficient matrix and $\mathbf{b} \in \mathbf{R}^3$ is the given right-hand side of the system.

**(a)** Use your MATLAB function files to generate a random $3 \times 5$ matrix $A$, a random vector $\mathbf{b} \in \mathbf{R}^3$ and the reduced row echelon form $R$ of the augmented matrix $[A \;\; \mathbf{b}]$:

```
A = rmat(3, 5), b = rvect(3), R = rref([A  b])
```

To get the reduced row echelon form $S = \text{rref}(A)$ just remove the last column from $R$:

```
S = R(:,1:5)
```

(Note the use of the colon operator to select columns 1 to 5 of $R$). Check by MATLAB that `S = rref(A)`; then write answers to the following:

*(i)* Which columns of $S$ are the *pivot columns*?

*(ii)* Which variables $x_i$ are the free variables?

*(iii)* What is the *rank* of $R$?

*(iv)* What is the *rank* of $A$?

*(v)* What is the *nullity* of $A$?

*(vi)* Why does the equation $A\mathbf{x} = \mathbf{b}$ have a solution?

**(b)** Use MATLAB to calculate `x = [R(:,6); 0; 0]`.

*(i)* Explain by a hand calculation (using symbolic algebra but not any arithmetic computations) why $\mathbf{x}$ must satisfy the equation $A\mathbf{x} = \mathbf{b}$.

Confirm by MATLAB that $A\mathbf{x} - \mathbf{b}$ is (almost) zero.

**IMPORTANT NOTE:** Because of the finite precision of computer arithmetic and roundoff error, vectors or matrices that are zero (theoretically) may appear in MATLAB in exponential form such as `1.0 e-15 ***` (where *** are numbers between 0 and 1). This means that each component is less than $10^{-15}$ in absolute value, so it can be treated as zero (numerically) in comparison to numbers on the order of 1 in size. Whenever you are asked to verify by MATLAB that two matrices or vectors are equal, calculate their difference and use this meaning of "zero". (Here answers of the form `1.0 e-14 ***` and `1.0 e-13 ***` would also be considered as "zero".)

**(c)** Use MATLAB to calculate `u = [-S(:,4); 1; 0],    v = [-S(:,5);0;1]`.

*(i)* Explain by a hand calculation why $\mathbf{u}$ and $\mathbf{v}$ are the *special solutions* to $A\mathbf{x} = \mathbf{0}$.

(see page 73 of the text). Confirm by calculating `S*u, A*u, S*v, A*v`. You should get vectors that are (approximately) zero. Now generate a random linear combination of $\mathbf{u}$ and $\mathbf{v}$ by the commands

```
s = rand(1),   t = rand(1), y = s*u + t*v
```

(Each occurrence of `rand(1)` generates a different random coefficient).

*(ii)* What properties of matrix and vector algebra ensure that $A\mathbf{y} = 0$?

Confirm by a MATLAB calculation that $A\mathbf{y}$ is approximately zero.

**(d)** Use MATLAB to calculate `z = x + y` .

*(i)* What properties of matrix and vector algebra imply that $A\mathbf{z} = \mathbf{b}$?

Confirm by a MATLAB calculation that $A\mathbf{z} - \mathbf{b}$ is approximately zero.

## Question 2. Spanning Sets and Linear Independence

Generate four random vectors in $\mathbf{R}^3$ by the command

```
u1 = rvect(3), u2 = rvect(3), u3 = rvect(3), u4 = rvect(3)
```

Use these vectors in the following.

**(a)** Consider the set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. To determine whether $\mathcal{S}$ is linearly independent, form the matrix $A$ with the vectors from $\mathcal{S}$ as columns and calculate its reduced row echelon form:

```
A = [u1 u2 u3],   R = rref(A)
```

Write comments that answer the following questions:

*(i)* How many free variables does the equation $A\mathbf{x} = \mathbf{0}$ have?

*(ii)* Is the set $\mathcal{S}$ linearly independent or linearly dependent? Why?

*(iii)* What is the rank of $A$?

*(iv)* What is the nullity of $A$?

**(b)** Consider the set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$. To determine whether $\mathcal{S}$ is linearly independent, form the matrix $A$ with the vectors from $\mathcal{S}$ as columns and calculate its reduced row echelon form:

```
A = [u1 u2 u3 u4],   R = rref(A)
```

(use the up-arrow key $\uparrow$ to save typing). Write comments that answer the following questions:

*(i)* How many free variables does the equation $A\mathbf{x} = \mathbf{0}$ have?

*(ii)* Is the set $\mathcal{S}$ linearly independent or linearly dependent? Why?

*(iii)* What is the rank of $A$?

*(iv)* What is the nullity of $A$?

**(c)** Take an arbitrary linear combination $\mathbf{v} = s\mathbf{u}_1 + t\mathbf{u}_2$:

```
v = rand(1)*u1 + rand(1)*u2
```

Consider the set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$. Is $\mathcal{S}$ linearly independent or linearly dependent? Answer first *without calculation* (give reasons for your answer). Then check by MATLAB using the method of part **(a)**.

## Question 3. Matrix Multiplication

For this question generate random matrices and a random vector:

```
A = rmat(2,3), B = rmat(3, 4), C = rmat(4,3), v = rvect(4)
```

To obtain the product $AB$ of the matrices $A$ and $B$ using MATLAB, you must type `A*B`. This is only defined when the number of columns of $A$ is the same as the number of rows of $B$ (you will get an error message when the matrix sizes are not compatible).

**(a) Associativity:** The product $AB$ is defined uniquely by the property that $A$ applied to a vector $\mathbf{u} = B\mathbf{v}$ is the same as the matrix $AB$ applied to the vector $\mathbf{v}$, for every vector $\mathbf{v}$ of the correct size. Verify this by calculating

```
u = B*v, A*u, D = A*B, D*v
```

This property implies the *associativity* of matrix multiplication: $A(BC) = (AB)C$. Verify this for the matrices $A, B, C$ that you have generated.

**(b) Matrices Are Not Numbers:** Matrix algebra has many similarities to (ordinary) algebra of numbers, but there are important differences (which are the mathematical source of the differences between classical mechanics and quantum mechanics in physics). Here are some examples. Generate matrices

```
A = [0  1; 0  0],  B = [0  0; 1  0],  C = [0  1; 1  0]
```

Use these matrices and MATLAB calculations to answer the following:

*(i)* Is $AB = BA$? Is $(A + B)^2 = A^2 + 2AB + B^2$? (Note that both of these equations would be true if $A$ and $B$ were numbers instead of matrices.)

*(ii)* Calculate $A^2$. If $A$ were a number instead of a matrix, what would the value of $A^2$ tell you about the value of $A$? Is this conclusion valid for matrices?

*(iii)* Calculate $AC$ and compare it with $AB$. If $A, B, C$ were numbers with $A \neq 0$, what would you conclude about $B$ and $C$ from this calculation? Is this conclusion valid for matrices?

## Question 4. Adjacency Matrices

Read the material on Adjacency Matrices in **Section 2.2**, pp. 103–106 in the text and work through Practice Problem #3 on page 108. The following questions refer to Exercise #18 on page 112 of the text. Be careful when you copy the $6 \times 6$ matrix $A$ into your MATLAB workspace. Before answering the questions, draw (by hand) an oriented graph with six vertices labeled 1–6 (one for each person); put an arrow from vertex $i$ to vertex $j$ when $a_{ij} = 1$. Include this graph with your write-up. Explain how you get your answers.

**(a)** Answer part (f) of Exercise #18.

**(b)** Answer part (g) of Exercise #18.

**(c)** Answer part (h) of Exercise #18.

**(d)** Is there an integer $k$ so that when $n \geq k$, *every* entry of $A^n$ not in column 5 is positive? Use MATLAB to explore this question. Then give an algebraic proof of your answer using properties of matrix multiplication.

**Final Editing of Lab Write-up:**

After you have worked through all the parts of the lab assignment, edit your diary file. Include the MATLAB calculations, but remove errors that you made in entering commands and remove other material that is not directly related to the questions, as you did for Lab 1.