BRIEF GUIDE TO COÖRDINATES AND MATRICES

References to Fraleigh & Beauregard, for short F & B, are references to John B. Fraleigh and Raymond A. Beauregard, *Linear Algebra,* 3rd ed. (1995), Addison-Wesley, ISBN# 0-201-52675-1. References to Blanchard, Devaney & Hall, for short BD&H, are references to Paul Blanchard, Robert L. Devaney & Glen R. Hall, *Differential Equations*, Brooks/Cole (1997), ISBN# 0-534-34550-6.

1.0. Coördinates: If V is an n-dimensional vector space and $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a(n ordered) basis of V, then the **coördinate vector**, or loosely the **coördinates of a vector $\mathbf{v} \in V$ relative to** B is/are the (column) vector (or its entries) $[r_1, \ldots, r_n]^T \in \mathbb{R}^n$ whose entries are the (uniquely determined) scalars for which $\mathbf{v} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$. See F & B, p. 205, who denote the row version of this vector by $\mathbf{v}_B$. (We shall largely eschew that notation here.) It follows from the uniqueness of the representation of elements of V as linear combinations of the elements of B that for any scalar $s \in \mathbb{R}$ the coördinates of $s\mathbf{v}$ are given by $[sr_1, \ldots, sr_n]^T \in \mathbb{R}^n$, and that if $\mathbf{v}$ and $\mathbf{w}$ are two vectors for which $\mathbf{v} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$ and $\mathbf{w} = s_1\mathbf{b}_1 + \cdots + s_n\mathbf{b}_n$ respectively, then "addition vertically"

$$\begin{aligned} \mathbf{v} &= r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n \\ \mathbf{w} &= s_1\mathbf{b}_1 + \cdots + s_n\mathbf{b}_n \\ \mathbf{v} + \mathbf{w} &= (r_1 + s_1)\mathbf{b}_1 + \cdots + (r_n + s_n)\mathbf{b}_n \end{aligned}$$

shows that the coördinates of $\mathbf{v} + \mathbf{w}$ are $[r_1 + s_1, \ldots, r_n + s_n]^T$. This definition agrees with customary usage in the case where V is $\mathbb{R}^n$ and the basis B is the "standard basis" $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ whose j-th element is the j-th standard basis vector $[0, \ldots, 1, \ldots, 0]^T$, *i.e.*, the vector of 0's and 1's in which the unique "1" is in the j-th row. Thus once a basis has been chosen, the vector-space operations in V behave exactly like the vector-space operations on their coördinate vectors relative to B in $\mathbb{R}^n$. This is the **isomorphism between V and $\mathbb{R}^n$** discussed by F & B on pp. 221–223 and in Theorem 3.9. Note that this isomorphism depends on the choice of B, so that even if V *is* $\mathbb{R}^n$ this isomorphism may not be the "identity." The consequences of this fact are discussed in the next §.

1.1. Coördinates with respect to non-standard bases in $\mathbb{R}^n$: If $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a(n ordered) basis of $\mathbb{R}^n$ that is not the standard basis, then the coördinates of a vector $\mathbf{v} = [x_1, \ldots, x_n]^T$ with respect to B will not be the x_i's. However, they are related to the x_i's as follows: if we also denote by B the matrix whose j-th column is $\mathbf{b}_j$, then B is an $n \times n$ matrix and $\mathbf{b}_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$. If $[r_1, \ldots, r_n]^T \in \mathbb{R}^n$ are the coördinates of $\mathbf{v}$ with respect to the basis B, we then have the two representations

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{v} = \sum_{j=1}^{n} r_j \mathbf{b_j} = \begin{bmatrix} \sum_{j=1}^{n} r_j b_{1j} \\ \vdots \\ \sum_{j=1}^{n} r_j b_{nj} \end{bmatrix} = B \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

for $\mathbf{v}$, where the fourth equal-sign holds by the definition of matrix multiplication. Comparing the extreme l. h. and r. h. sides, we see that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$B^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} .$$

In words: **to find the coördinates of v with respect to the basis of $\mathbb{R}^n$ whose elements are the columns of the matrix B, simply compute $B^{-1}\mathbf{v}$.**

On p. 207, F & B tell the reader to compute these coördinates by writing the "augmented matrix" $[B|\mathbf{v}]$ and row-reducing it to the form $[I_n|\mathbf{v}_B]$. Since this is the standard Gauss-Jordan way to solve the (system of) equation(s) $B \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *where the r's are regarded as unknown and the x's as known,* of course this is equivalent to saying $\mathbf{v}_B = B^{-1}\mathbf{v}$, and **this method is better than computing B^{-1} for producing numbers,** because it is more efficient computationally; **but it is not as good when one has to derive formulas and/or think.**

1.2. Coördinates with respect to a basis B of a k-dimensional subspace $V \subseteq \mathbb{R}^n$: Exactly the same computations made above show that if $B = (\mathbf{b}_1, \ldots, \mathbf{b}_k)$ is a basis of a k-dimensional subspace V of $\mathbb{R}^n$, and if B also denotes the matrix whose j-th column is $\mathbf{b}_j$, then for any element $[x_1, \ldots, x_n]^T = \mathbf{v}$ of V, we must have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix} .$$

Since B has only k columns—it is an $n \times k$ matrix, and $k \leq n$, so it is in general not a square matrix—there is in general no matrix B^{-1}, so we cannot just apply B^{-1} to both sides of this equation to find the r_i's. However, we *can* apply B^T (which is $k \times n$) to both sides of this equation, and it can be shown[(1)] that the $k \times k$ matrix $B^T B$ must be invertible if the columns of B are linearly independent. Consequently, we can write

$$\begin{aligned} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= B \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix} \\ B^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= B^T B \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix} \\ (B^T B)^{-1} B^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix} \end{aligned}$$

to give a formula for the r_i's.

In practice—for numerical calculation—one solves the equations

$$B^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (B^T B) \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}$$

by writing the "augmented matrix" $[B|\mathbf{v}]$—which does not have a square B—and then multiplying it by B^T on the left to give an "augmented matrix" $[B^T B|B^T\mathbf{v}]$ corresponding to a $k \times k$ system of equations. This is then solved by row-reducing it to $[I_k|\mathbf{v}_B]$.

(1) The Spring 2000 main web page for Math 250:04 has a link to notes "Basis expansions and orthogonal projections" containing a version of the details.

This is equivalent to multiplying $B^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ by $(B^TB)^{-1}$ but represents less numerical work. It is also a more stable numerical procedure when floating-point calculations are involved.

If B happens to be $n \times n$ (because $V = \mathbb{R}^n$) then $(B^TB)^{-1}B^T = B^{-1}(B^T)^{-1}B^T = B^{-1}$ and this process reduces to the one considered in **1.1** above. It can be shown that if $\mathbf{v} \in \mathbb{R}^n$ does *not* belong to V, then the process that we just described produces the coördinates with respect to B of the **orthogonal projection of v** on V. Details—for anyone who might be interested in them—can be found in F & B, §6.4, pp. 360 ff. This fact implies that the *vector* orthogonal projection of $\mathbf{v} \in \mathbb{R}^n$ on V is given(2) by $B(B^TB)^{-1}B^T\mathbf{v}$.

2.0. The matrix of a linear transformation: In their §2.3, pp. 142 ff., F & B produce the **standard matrix representation** of a given linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$; this is the matrix

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

and it has the property that for each $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$. F & B give the details. Note that if the entries in $A = [a_{ij}]$ are indexed in the usual way, then the j-th column of the matrix A, which is $T(\mathbf{e}_j)$, can be written as

$$T(\mathbf{e}_j) = a_{1j}\mathbf{e}_1 + a_{2j}\mathbf{e}_2 + \cdots + a_{mj}\mathbf{e}_m = \sum_{i=1}^{m} a_{ij}\mathbf{e}_i\ . \tag{$*$}$$

In the case of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we have $n = m$ and A is square.

2.1. The matrix of a linear transformation relative to a basis: It is easy to adapt the construction just given(3) to linear transformations $T : V \to V$ from one "abstract" vector space to itself: however, since there is no distinguished "standard" basis in such a V, we have to talk about **the matrix**, or **matrix representation**, of T **relative to a basis** $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of V. We simply replace the $\mathbf{e}_i$'s of the standard construction in the preceding paragraph by the $\mathbf{b}_i$'s that we have, so that the matrix we want is $A = [a_{ij}]$ where

$$T(\mathbf{b}_j) = a_{1j}\mathbf{b}_1 + a_{2j}\mathbf{b}_2 + \cdots + a_{nj}\mathbf{b}_n = \sum_{i=1}^{n} a_{ij}\mathbf{b}_i\ . \tag{$**$}$$

That is, the j-th column of A is the coördinate vector—with respect to the basis B—of $T(\mathbf{b}_j)$. Using this defining relation, it is easy to check that if $U : V \to V$ is a(nother) linear transformation whose matrix representation is $C = [c_{ij}]$, then $U \circ T : \mathbf{v} \to U(T(\mathbf{v}))$ has the matrix representation CA. Indeed,

$$\begin{aligned} T(\mathbf{b}_j) = a_{1j}\mathbf{b}_1 + a_{2j}\mathbf{b}_2 + \cdots + a_{nj}\mathbf{b}_n &= \sum_{k=1}^{n} a_{kj}\mathbf{b}_k \\ (U \circ T)(\mathbf{b}_j) = U(T(\mathbf{b}_j)) &= \sum_{k=1}^{n} a_{kj}U(\mathbf{b}_k) = \sum_{k=1}^{n} a_{kj}\left(\sum_{i=1}^{n} c_{ik}\mathbf{b}_i\right) \\ &= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{kj}c_{ik}\right)\mathbf{b}_i = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} c_{ik}a_{kj}\right)\mathbf{b}_i \end{aligned}$$

so the matrix representation of $U \circ T$ is given by the matrix $\left(\sum_{k=1}^{n} c_{ik}a_{kj}\right)$, which is just the matrix product CA (in that order, the same order in which T and U are composed).

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(2) Note that the apparent cancellation in the expression $B(B^TB)^{-1}B^T$ does not occur if the matrix B is not square.

(3) In fact, one can adapt it for linear transformations $T{:}V{\to}V'$ from one "abstract" vector space to another, by selecting a basis B for V and a basis B' for V'. The details are in F & B, pp. 223–225. However, the case of a linear transformation of a vector space to itself is the most important in applications, and conventionally one uses the same basis at "both ends" of T.

2.2. Changing bases and changing coördinates: Suppose V is an n-dimensional vector space and $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ and $B' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ are two bases of V; then a vector $\mathbf{v} \in V$ has two competing vectors of coördinates: its coördinates $[r_1, \dots, r_n]^T \in \mathbb{R}^n$ relative to B, whose entries are the (uniquely determined) scalars for which $\mathbf{v} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$, and its coördinates $[s_1, \dots, s_n]^T \in \mathbb{R}^n$ relative to B', whose entries are the (uniquely determined) scalars for which $\mathbf{v} = s_1\mathbf{b}'_1 + \cdots + s_n\mathbf{b}'_n$. It is not difficult to see how to convert one vector of coördinates to the other. Each vector $\mathbf{b}'_j$ can be written in a unique way as a linear combination of the $\mathbf{b}_i$'s:

$$\mathbf{b}'_j = \sum_{i=1}^{n} c_{ij}\mathbf{b}_i \ .$$

Plugging this into the coördinate representation of $\mathbf{v}$ relative to B' gives

$$r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n = \mathbf{v} = \sum_{j=1}^{n} s_j\mathbf{b}'_j = \sum_{j=1}^{n} s_j \left(\sum_{i=1}^{n} c_{ij}\mathbf{b}_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}s_j\right)\mathbf{b}_i \ ;$$

comparing the coördinates of $\mathbf{b}_i$ on the extreme l. h. and r. h. sides of this equation gives

$$r_i = \sum_{j=1}^{n} c_{ij}s_j \quad \text{for } j = 1, \dots, n$$

and those scalar equations are equivalent to the single vector-matrix equation

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} .$$

The situation is symmetrical with respect to the rôles of B and B', so we can play the same game backwards: each vector $\mathbf{b}_j$ can be written in a unique way as a linear combination of the $\mathbf{b}_i$'s:

$$\mathbf{b}_j = \sum_{i=1}^{n} d_{ij}\mathbf{b}'_i \ .$$

Plugging this into the coördinate representation of $\mathbf{v}$ relative to B' gives

$$s_1\mathbf{b}'_1 + \cdots + s_n\mathbf{b}'_n = \mathbf{v} = \sum_{j=1}^{n} r_j\mathbf{b}_j = \sum_{j=1}^{n} r_j \left(\sum_{i=1}^{n} d_{ij}\mathbf{b}'_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}r_j\right)\mathbf{b}'_i \ ;$$

comparing the coördinates of $\mathbf{b}'_i$ on the extreme l. h. and r. h. sides of this equation gives

$$s_i = \sum_{j=1}^{n} d_{ij}r_j \quad \text{for } j = 1, \dots, n$$

and those scalar equations are equivalent to the single vector-matrix equation

$$\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \cdots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} .$$

It follows easily that $[d_{ij}][c_{ij}] = I_n$ and $[c_{ij}][d_{ij}] = I_n$ as matrix products: the two "change-of-basis matrices" are each other's inverses. Thus we have a choice of writing

$$\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \cdots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

as the formula that **converts from the coördinates relative to B to the coördinates relative to B'**. For numerical work, it is usually easier to solve the system of equations

$$\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

by the usual augmented-matrix/Gauss-Jordan reduction technique than to invert the matrix $[c_{ij}]$. But notice that this is an **inverse**[(4)] relation: if $[c_{ij}]$ is the matrix that writes B' in terms of B, then $[c_{ij}]^{-1}$ converts coördinates with respect to B into coördinates with respect to B'.

2.3. Changing bases and changing the matrix of a linear transformation: This follows the same pattern as changing coördinates. If $T : V \to V$ is a linear transformation, $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $B' = (\mathbf{b}'_1, \ldots, \mathbf{b}'_n)$ are two bases of V, the matrix representation of T relative to B is $[a_{ij}]$ and the matrices that write B and B' in terms of each other are $[c_{ij}]$ and $[d_{ij}]$ respectively, so $\mathbf{b}'_j = \sum_{i=1}^{n} c_{ij}\mathbf{b}_i$ and $\mathbf{b}_j = \sum_{i=1}^{n} d_{ij}\mathbf{b}'_i$ respectively, then

$$\begin{aligned} T(\mathbf{b}'_j) &= T\left(\sum_{k=1}^{n} c_{kj}\mathbf{b}_k\right) = \sum_{k=1}^{n} c_{kj}T(\mathbf{b}_k) = \sum_{k=1}^{n} c_{kj}\left(\sum_{\ell=1}^{n} a_{\ell k}\mathbf{b}_\ell\right) \\ &= \sum_{k=1}^{n} c_{kj}\left[\sum_{\ell=1}^{n} a_{\ell k}\left(\sum_{i=1}^{n} d_{i\ell}\mathbf{b}'_i\right)\right] = \sum_{i=1}^{n}\left(\sum_{\ell=1}^{n}\sum_{k=1}^{n} d_{i\ell}a_{\ell k}c_{kj}\right)\mathbf{b}'_i\,. \end{aligned} \tag{\#}$$

Since (#) has the form $T(\mathbf{b}'_j) = \sum_{i=1}^{n} g_{ij}\mathbf{b}'_i$, the coefficients that occur in its r. h. s. *must* be the entries in the matrix of T relative to B'. By the definition of matrix multiplication, the double sum $\left(\sum_{\ell=1}^{n}\sum_{k=1}^{n} d_{i\ell}a_{\ell k}c_{kj}\right)$ is the ij-indexed entry in the matrix $[d_{ij}][a_{ij}][c_{ij}] = [c_{ij}]^{-1}[a_{ij}][c_{ij}]$, so we now know:

> **If $T : V \to V$ is a linear transformation, $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $B' = (\mathbf{b}'_1, \ldots, \mathbf{b}'_n)$ are two bases of V, the matrix representation of T relative to B is $[a_{ij}]$ and the matrix that writes B' in terms of B is $[c_{ij}]$, so $\mathbf{b}'_j = \sum_{i=1}^{n} c_{ij}\mathbf{b}_i$, then the matrix of T relative to B' is $[c_{ij}]^{-1}[a_{ij}][c_{ij}]$.**

Consider the particular case in which $V = \mathbb{R}^n$, the rôle of B is played by the standard basis, and the rôle of B' by a non-standard basis. We are accustomed to call the new basis by the name B, which is a bit confusing. However, in this case the matrix we called "$[c_{ij}]$" in the preceding paragraph is then just the matrix $B = \begin{bmatrix} | & \cdots & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & \cdots & | \end{bmatrix} = [b_{ij}]$ whose j-th column is the vector $\mathbf{b}_j$. So the general formula we just derived becomes

(4) The word *contravariant* is also sometimes used to describe a relation that behaves this way.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation whose standard matrix representation is $A = [a_{ij}]$, $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a(nother) basis of $\mathbb{R}^n$, and $B = [b_{ij}]$ also denotes the $n \times n$ matrix whose j-th column is $\mathbf{b}_j$, then the matrix of T relative to B is $[b_{ij}]^{-1}[a_{ij}][b_{ij}] = B^{-1}AB$.

Reversing the rôles of the standard basis and the new basis in the result given above, we see that

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation whose matrix representation is $A = [a_{ij}]$ relative to a (non-standard) basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of $\mathbb{R}^n$, and $B = [b_{ij}]$ also denotes the $n \times n$ matrix whose j-th column is $\mathbf{b}_j$, then the matrix of T relative to the standard basis of $\mathbb{R}^n$ is $[b_{ij}][a_{ij}][b_{ij}]^{-1} = BAB^{-1}$.

2.4. Diagonalization and diagonalizable matrices: The most important case of what we have just done is probably the one that occurs when $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation for which there is a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of $\mathbb{R}^n$ whose elements are **eigenvectors of** T, *i.e.*, such that for each $\mathbf{b}_j$ there is a scalar λ_j for which $T\mathbf{b}_j = \lambda_j \mathbf{b}_j$. Such linear transformations are called **diagonalizable**, since the matrix of T relative to this basis is easily seen to have the particularly simple **diagonal form**

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} .$$

An $n \times n$ matrix is then called **diagonalizable** if its corresponding linear transformation $T : \mathbf{x} \to A\mathbf{x}$ of $\mathbb{R}^n$ is diagonalizable (with respect to *some* basis, in general not the standard basis.) The two important results we listed in **2.3** above specialize in this case to

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation whose standard matrix representation is $A = [a_{ij}]$, and if $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a basis of $\mathbb{R}^n$ such that for each $j = 1, \ldots, n$ there is a scalar λ_j for which $T(\mathbf{b}_j) = \lambda_j \mathbf{b}_j$, and $B = [b_{ij}]$ also denotes the $n \times n$ matrix whose j-th column is $\mathbf{b}_j$, then the matrix of T relative to B is

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = B^{-1}AB . \tag{!}$$

Reversing the rôles of the standard basis and the new basis in the result given above, or simply multiplying both sides of the equation (!) by B on the left and B^{-1} on the right, we see that

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with the property that there is a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of $\mathbb{R}^n$ such that for each $j = 1, \ldots, n$ there is a scalar λ_j for which $T(\mathbf{b}_j) = \lambda_j \mathbf{b}_j$, and if $B = [b_{ij}]$ also denotes the $n \times n$ matrix whose j-th column is $\mathbf{b}_j$, then the matrix of T relative to the standard basis—*i.e.*, the standard matrix representation of T—is the matrix A given by

$$A = B \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} B^{-1} . \tag{!!}$$

3.1. Differential-equation specialties[5]: For any natural number k we can compute the k-th power of a diagonalizable matrix A in a particularly easy way: it is trivial to verify that the k-th power of a diagonal matrix is given by simply raising the diagonal elements to the k-th power:

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

and this makes computation of the k-th power of a *diagonalizable* matrix A almost as easy:

$$\begin{aligned} A^k &= \left(B \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} B^{-1} \right)^k \\ &= \underbrace{B \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} B^{-1} B \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} B^{-1} \cdots B \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} B^{-1}}_{k \text{ factors}} \\ &= B \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{k \text{ factors}} B^{-1} \\ &= B \left(\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \right)^k B^{-1} = B \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} B^{-1} \end{aligned}$$

because all the intermediate products $B^{-1}B$ equal the identity. Assuming that the interchange of algebraic and limiting operations is justified, we can then write for $t \in \mathbb{R}$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} B^{-1} = B \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_1 t)^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda_2 t)^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{(\lambda_n t)^k}{k!} \end{bmatrix} B^{-1}$$

[5] Most of this material will be irrelevant for Math 250 students, but Math 252 students may find that it has survival value.

and since we know the sums of the series $\sum_{k=0}^{\infty} \frac{(\lambda_j t)^k}{k!} = e^{\lambda_j t}$ on the diagonals of the matrix, we can simply write

$$e^{tA} = B \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} B^{-1} . \qquad (\$)$$

In some sense this formula "rationalizes" the method of "straight-line solutions" or eigenvectors of BD&H, §§3.2 and 3.3. There one solved the 2×2 linear homogeneous system $\mathbf{Y}' = A\mathbf{Y}$ by finding eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$ of A corresponding to its two distinct real eigenvalues λ_1 and λ_2, and observing that each function $e^{\lambda_j t}\mathbf{V}_j$ satisfied $\mathbf{Y}' = A\mathbf{Y}$. One then observed, since $\{\mathbf{V}_1, \mathbf{V}_2\}$ was a basis of $\mathbb{R}^2$, that given any initial position vector $\mathbf{Y}_0$ one could find constants k_1 and k_2 for which $\mathbf{Y}_0 = k_1\mathbf{V}_1 + k_2\mathbf{V}_2$. It then followed that $\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$ satisfied both the differential equation and the initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$. From the standpoint of the equation ($), we are simply writing

$$e^{tA} = B \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} B^{-1}$$

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 = B \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} B^{-1}\mathbf{Y}_0$$

where B is the matrix whose j-th column is $\mathbf{V}_j$, $j = 1, 2$; indeed, the equation $\mathbf{Y}_0 = k_1\mathbf{v}_1 + k_2\mathbf{V}_2$ is solved by writing it in vector-matrix form as $\mathbf{Y}_0 = [\mathbf{V}_1\mathbf{V}_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = B \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$, so that $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = B^{-1}\mathbf{Y}_0$ and thus

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 = B \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = B \begin{bmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2} t \end{bmatrix} = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2 .$$

Note what is going on here. If one uses coördinates with respect to the basis $\{\mathbf{V}_1, \mathbf{V}_2\}$ rather than the standard coördinates that come with the standard basis, then the matrix of $\mathbf{Y}' = A\mathbf{Y}$ is diagonal: if z_1, z_2 are the coördinates with respect to that basis, the vector DE $\mathbf{Y}' = A\mathbf{Y}$ is equivalent to the two *uncoupled* scalar DEs $z_1' = \lambda_1 z_1$ and $z_2' = \lambda_2 z_2$. The "straight-line solutions" travel along the z_1- and z_2-axes; other trajectories have the form $z_2 = \pm(\text{const.}) \cdot (z_1)^{\lambda_2/\lambda_1}$, as in BD&H's systems on pp. 251–252 and p. 255. Only one's insistence on using the standard basis in which to write the equation, instead of using the basis of eigenvectors of A that the system "wants," is responsible for "distorted" pictures like Fig. 3.14 and Fig. 3.16.

3.2 Equal eigenvalues, $n = 2$: In higher dimensions the case of "equal eigenvalues" can get complicated; fortunately, it undergoes remarkable simplification[(6)] in dimension 2. "Equal eigenvalues" is a misnomer: if there is a basis $\{\mathbf{V}_1, \mathbf{V}_2\}$ of $\mathbb{R}^2$ for which $A\mathbf{V}_1 = \lambda\mathbf{V}_1$ and $\mathbf{V}_2 = \lambda\mathbf{V}_2$ (same λ), then the matrix of A relative to the basis $\{\mathbf{V}_1, \mathbf{V}_2\}$ is $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, A equals multiplication by the scalar λ, and $\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$ solves the initial value problem for $\mathbf{Y}' = A\mathbf{Y}$. The problem comes with a matrix like $\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$, whose characteristic polynomial $\det\left(\begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}\right) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$ has only one (double) root but which

(6) I first heard this observation from Professor Sontag, and the exposition here is similar to that in his notes of a few years ago, available from the "general Math 252 web page."

has only one eigenvector. (The eigenvectors belonging to $\lambda = 2$ are solutions of $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{0}$, and there is only one [up to scalar multiples], namely $\mathbf{V}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.) However, if one chooses *any* vector $\mathbf{V}_2$ linearly independent of the single eigenvector $\mathbf{V}_1$ belonging to the single eigenvalue—let's call it λ_1—then the two vectors $\{\mathbf{V}_1, \mathbf{V}_2\}$ will necessarily form a basis of $\mathbb{R}^2$, and because

$$A\mathbf{V}_1 = \lambda_1 \mathbf{V}_1 \quad \text{and} \quad A\mathbf{V}_2 = \alpha \mathbf{V}_1 + \beta \mathbf{V}_2$$

for *some* scalars α, β, the matrix of A relative to this basis will have the form $\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \beta \end{bmatrix}$. The calculation

$$\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \beta \end{bmatrix} = B^{-1}AB$$

$$\begin{aligned}(\lambda - \lambda_1)(\lambda - \beta) &= \det\left(\lambda I_2 - \begin{bmatrix} \lambda_1 & \alpha \\ 0 & \beta \end{bmatrix}\right) = \det(\lambda I_2 - B^{-1}AB) = \det(\lambda B^{-1}B - B^{-1}AB) \\ &= (\det B^{-1}) \det(\lambda I_2 - A)(\det B) = \det(\lambda I_2 - A) = (\lambda - \lambda_1)^2\end{aligned}$$

then shows that in fact $\beta = \lambda_1$ must hold, else the characteristic polynomial of A would have had two distinct roots—but that is not the case we are now considering. So the matrix of A with respect to the basis B has the form $\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_1 \end{bmatrix} = \lambda_1 I_2 + \alpha N$, where N is the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $N^2 = 0$. It follows that

$$\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_1 \end{bmatrix}^k = (\lambda_1 I_2 + \alpha N)^k = \lambda_1^k I_2 + k\lambda_1^{k-1}\alpha N + (\cdots)N^2 = \lambda_1^k I_2 + \lambda_1^{k-1}\alpha N$$

since all the terms containing N^ℓ for $\ell \geq 2$ will be zero, and thus

$$\exp\left(t \begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_1 \end{bmatrix}\right) = \sum_{k=0}^{\infty} \frac{[t(\lambda_1 I_2 + \alpha N)]^k}{k!} = \sum_{k=0}^{\infty} \frac{(t\lambda_1)^k}{k!} I_2 + \sum_{k=0}^{\infty} \frac{t^k \cdot k\lambda_1^{k-1}}{k!} \alpha N = e^{\lambda_1 t} I_2 + \alpha t e^{\lambda_1 t} N$$

$$\exp(tA) = B \exp\left(t \begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_1 \end{bmatrix}\right) B^{-1} = e^{\lambda_1 t} I_2 + \alpha t e^{\lambda_1 t} BNB^{-1}\,.$$

Continuing with $A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$, for example, we have $\mathbf{V}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and we might take $\mathbf{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; then $A\mathbf{V}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \mathbf{V}_1 + 2\mathbf{V}_2$. The matrix of A relative to this basis is then $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$; $\alpha = 1$ and we have $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, so $BNB^{-1} = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$. Finally, then,

$$e^{tA} = e^{2t} I_2 + t e^{2t} \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{bmatrix}\,.$$

It is easy to verify that the derivative of $e^{2t} \begin{bmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{bmatrix}$ is in fact $\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} e^{2t} \begin{bmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{bmatrix}$.