First-order Linear Systems of Differential Equations

as a concrete example of an abstract vector space

The solutions of these systems form a nice example of "abstract" finite-dimensional vector spaces that arise in considering a concrete problem. For people who have not yet had a course in differential equations, or who may need a refresher, here is a brief synopsis of the subject, with some proofs.

A **homogeneous linear system of first-order differential equations** is a system of equations of the form

$$\begin{aligned}\frac{dy_1}{dt} &= a_{11}(t)y_1(t) + \cdots + a_{1n}(t)y_n(t)\\ &\cdots\\ &\cdots\\ &\cdots\\ \frac{dy_n}{dt} &= a_{n1}(t)y_1(t) + \cdots + a_{nn}(t)y_n(t)\end{aligned}$$

where the **coefficients** or **coefficient functions** $\{a_{ij}(t)\}_{i,j=1}^{n,n}$ are continuous functions on some interval $(a,b) \subseteq \mathbb{R}$. A **solution** of the system on some interval $(c,d) \subseteq (a,b)$ is a family $\{y_i(t)\}_{i=1}^{n}$ of (continuously) differentiable functions for which the equations hold identically on (c,d). A simple example is furnished by the system $dy_1/dt = -y_2(t)$, $dy_2/dt = y_1(t)$, a solution of which is given, for any choice of the constants A and φ, by $y_1(t) = A\cos(t-\varphi)$, $y_2(t) = A\sin(t-\varphi)$. It is natural to use matrix- and vector-notation for these objects: introducing the vector and matrix functions

$$\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} \text{ and } A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix},$$

and performing differentiation entry-by-entry (or coördinate-by-coördinate), we can write the system compactly in the form

$$\frac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)\ .$$

In many cases—and, indeed, in the only cases that we shall consider explicitly—the **matrix of coefficients** $A(t)$ is just a constant matrix A. In the example we gave above, we had a constant matrix of coefficients: the system had the form

$$\begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

While we are not going to take the time to prove the existence theorem for these systems (although it is not difficult, it requires the notion of **uniform convergence** of a sequence of functions on an interval, so it looks more like analysis ["advanced calculus"] than linear algebra), the following is a fact: if the coefficients in the system

$$\frac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)$$

are continuous functions on an interval $(a,b) \subseteq \mathbb{R}$, then given any $a < t_0 < b$ and any "initial position vector" $\mathbf{y}_0$, there exists a "solution vector" $\mathbf{Y}(t)$ of (continuously) differentiable functions defined for all $a < t < b$ and such that $\mathbf{Y}(t_0) = \mathbf{y}_0$, satisfying $\dfrac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)$ for all $a < t < b$. What we shall prove is that this solution is **unique**: if $\mathbf{Y}(t)$ and $\widehat{\mathbf{Y}}(t)$ are two such functions—so $\widehat{\mathbf{Y}}(t)$ is also defined for all $a < t < b$, $\widehat{\mathbf{Y}}(t_0) = \mathbf{y}_0$, and $\dfrac{d\widehat{\mathbf{Y}}(t)}{dt} = A(t)\widehat{\mathbf{Y}}(t)$ for all $a < t < b$—then in fact $\widehat{\mathbf{Y}}(t) = \mathbf{Y}(t)$ for all $a < t < b$. That proof will be given below, but it helps to make an observation that shows why one would be interested in these differential equations in the context of a course in (finite-dimensional) linear algebra.

The observation is that the set of vector-valued (continuously) differentiable functions on (a, b) that satisfy the equation $\frac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)$ for all $a < t < b$ is a vector space. To see this, all one has to do is let $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ be two functions that satisfy the equation, let a and b be two constants, and check that

$$\begin{aligned}\frac{d(a\mathbf{Y}_1 + b\mathbf{Y}_2)}{dt} &= a\,\frac{d\mathbf{Y}_1(t)}{dt} + b\,\frac{d\mathbf{Y}_2(t)}{dt} \\ &= a\,A(t)\mathbf{Y}_1(t) + b\,A(t)\mathbf{Y}_2(t) \\ &= A(t)\,[a\mathbf{Y}_1 + b\mathbf{Y}_2]\ .\end{aligned}$$

Every step follows from such well-known facts as the linearity of differentiation, the distributive law for matrix multiplication, and the fact that matrix multiplication commutes with the operation of multiplying by a constant. So if we have two solutions $\mathbf{Y}(t)$ and $\widehat{\mathbf{Y}}(t)$ of the differential equation, such that $\mathbf{Y}(t_0) = \widehat{\mathbf{Y}}(t_0)$ at some point t_0 of their domain, then $\mathbf{Z}(t) = \mathbf{Y}(t) - \widehat{\mathbf{Y}}(t)$ will also be a solution of the differential equation, and it will satisfy $\mathbf{Z}(t_0) = \mathbf{0}$. If we can prove that this forces $\mathbf{Z}(t) \equiv \mathbf{0}$—that $\mathbf{Z}(t)$ never changes—that will prove uniqueness of the solution determined by the "initial value" $\mathbf{y}_0$ that it is made to take at t_0.

So we shall now prove our uniqueness theorem. To do this it will essentially suffice to prove the following two lemmas. The first is based on the Cauchy-Buniâkovskiĭ-Schwarz inequality relating the absolute value of a dot product to the norms of the dot-factors (see Fraleigh & Beauregard, p. 24, inequality (10)).

Lemma: Let $A = [a_{ij}]$ be an $m \times n$ matrix and $\mathbf{v} = [v_1, \ldots, v_n]^T$ be an n-dimensional vector. If $\|A\|_2$ denotes the square root of the sum of the squares of the entries of A—in symbols, $\|A\|_2 = \sqrt{\sum_{i=1,\, j=1}^{m,n} a_{ij}^2}$—then the norm of the vector $A\mathbf{v}$ is bounded by $\|A\|_2\,\|\mathbf{v}\|$—in symbols, $\|A\mathbf{v}\| \le \|A\|_2\,\|\mathbf{v}\|$. ($\|A\|_2$ is called the (2-)**norm** of A.)

Proof. For $1 \le i \le m$ denote the i-th row of A, considered as an n-dimensional row vector, by $\mathbf{a}_i = [a_{i1}, \ldots, a_{in}]$. Then the definition of matrix multiplication says precisely that the vector $A\mathbf{v} = [\mathbf{a}_1 \bullet \mathbf{v}, \ldots, \mathbf{a}_m \bullet \mathbf{v}]^T$. For the i-th coöordinate of this vector, the Cauchy-Buniâkovskiĭ-Schwarz inequality gives the estimate $|\mathbf{a}_i \bullet \mathbf{v}| \le \|\mathbf{a}_i\|\,\|\mathbf{v}\|$. Squaring these estimates and adding, we have

$$\begin{aligned}(\mathbf{a}_1 \bullet \mathbf{v})^2 &\le \|\mathbf{a}_1\|^2\,\|\mathbf{v}\|^2 \\ &\cdots \\ &\cdots \\ &\cdots \\ (\mathbf{a}_m \bullet \mathbf{v})^2 &\le \|\mathbf{a}_m\|^2\,\|\mathbf{v}\|^2 \\ \sum_{i=1}^{m} (\mathbf{a}_i \bullet \mathbf{v})^2 = \|A\mathbf{v}\|^2 &\le \{\sum_{i=1}^{m} \|\mathbf{a}_i\|^2\}\,\|\mathbf{v}\|^2 \\ \|A\mathbf{v}\|^2 &\le \{\sum_{i=1}^{m}[\sum_{j=1}^{n} a_{ij}^2]\}\,\|\mathbf{v}\|^2 = \|A\|_2^2\,\|\mathbf{v}\|^2\ ,\end{aligned}$$

and the conclusion of the lemma follows by taking the square root of the last displayed formula above.

Lemma: Let $\mathbf{Z}(t)$ satisfy the equation

$$\frac{d\mathbf{Z}(t)}{dt} = A(t)\mathbf{Z}(t)$$

for all $c \le t \le d$, where the entries in the matrix $A(t)$ are continuous functions of t, and let t_0 lie in the interval $[c, d]$. Then there is a constant K such that for all $c \le t \le d$ the inequality

$$\|\mathbf{Z}(t)\| \le \|\mathbf{Z}(t_0)\| \cdot e^{K\,|t - t_0|}$$

holds. Thus, in particular, if $\mathbf{Z}(t_0) = \mathbf{0}$ at some $t_0 \in [c, d]$, then $\mathbf{Z}(t) \equiv \mathbf{0}$ for all $t \in [c, d]$.

Proof. First of all, we need the fact that since the entries in the matrix $A(t)$ are continuous functions of t on the interval $c \le t \le d$, so is the sum of their squares $\|A(t)\|_2^2$. Therefore, since any continuous function defined on a closed interval of the real line is bounded, there exists a constant $K \ge 0$ for which $\|A(t)\|_2 \le K$ holds for all $c \le t \le d$.† Let $\mathbf{Z}(t) = [z_1(t), \ldots, z_n(t)]^T$, and let $z(t) = \|\mathbf{Z}(t)\|^2 = z_1^2(t) + \cdots + z_n^2(t)$, the norm-squared of $\mathbf{Z}(t)$. Then

$$\begin{aligned}\frac{dz}{dt} &= 2\,z_1(t)\,\frac{dz_1}{dt} + \cdots + 2\,z_n(t)\,\frac{dz_n}{dt} = 2\,\mathbf{Z}(t) \bullet \frac{d\mathbf{Z}}{dt} \\ &= 2\,\mathbf{Z}(t) \bullet A(t)\mathbf{Z}(t)\ . \qquad (*)\end{aligned}$$

If we apply the Cauchy-Buni͡akovskiĭ-Schwarz inequality and then the first Lemma above to the r. h. s. of the relation $(*)$, we get the estimate

$$\begin{aligned}|\mathbf{Z}(t) \bullet A(t)\mathbf{Z}(t)| &\le \|\mathbf{Z}(t)\|\|A(t)\|_2\,\|\mathbf{Z}(t)\| \\ &\le K\,\|\mathbf{Z}(t)\|^2 \qquad (**)\end{aligned}$$

because $\|A(t)\|_2 \le K$ holds for all $t \in [c,d]$. Combining $(*)$ and $(**)$, we get

$$\begin{aligned}|\frac{dz}{dt}| \le 2K\,\|\mathbf{Z}(t)\|^2 = 2K\,z(t), \quad \text{or} \\ -2K\,z(t) \le \frac{dz}{dt} \le 2K\,z(t)\ . \qquad (\$)\end{aligned}$$

Now "linear differential inequalities" like ($) can be solved by pretty much the same method employed for linear differential equations: given the differential equation $dz/dt = az$, where a is a constant and $z(t)$ is to be a function of t, one multiplies the equation by the "integrating factor" e^{-at}, obtaining $e^{-at}\,dz/dt = e^{-at}az$, or $e^{-at}\,dz/dt - e^{-at}az = 0$, or $\dfrac{d}{dt}[e^{-at}z] = 0$. This last equation says that the function $e^{-at}z(t)$ is a constant, and one can find out the value of a constant function by evaluating it at any value of t: so one has $e^{-at}z(t) \equiv e^{-at_0}z(t_0)$, or $z(t) \equiv e^{a(t-t_0)}z(t_0)$. Similarly, one can multiply each "half" of the differential inequality ($) by an integrating factor. For the inequality $\dfrac{dz}{dt} \le 2K\,z(t)$, one multiplies by e^{-2Kt}, obtaining

$$\begin{aligned}e^{-2Kt}\,\frac{dz}{dt} &\le 2K\,e^{-2Kt}\,z(t) \\ e^{-2Kt}\,\frac{dz}{dt} - 2K\,e^{-2Kt}\,z(t) &\le 0 \\ \frac{d}{dt}[e^{-2Kt}\,z(t)] &\le 0\ .\end{aligned}$$

The derivative's being ≤ 0 signals that the function $e^{-2Kt}\,z(t)$ is a decreasing (*i.e.*, non-increasing) function and therefore for $t_0 \le t$ one has $e^{-2Kt}\,z(t) \le e^{-2Kt_0}\,z(t_0)$, or $z(t) \le e^{2K(t-t_0)}\,z(t_0)$. In a similar manner, but working on the "other half" $-2K\,z(t) \le \dfrac{dz}{dt}$ of the inequality, one shows that $z(t) \le e^{2K(t_0-t)}\,z(t_0)$ for $t \le t_0$. Thus both cases ($t \ge t_0$ and $t \le t_0$) are covered by the single formulation

$$\begin{aligned}z(t) &\le e^{2K|t-t_0|}\,z(t_0) \\ \|\mathbf{Z}(t)\|^2 &\le e^{2K|t-t_0|}\,\|\mathbf{Z}(t_0)\|^2 \\ \|\mathbf{Z}(t)\| &\le e^{K\,|t-t_0|}\,\|\mathbf{Z}(t_0)\|\end{aligned}$$

and that is the Lemma we set out to prove.

We now have the following uniqueness theorem.

† In many important cases, the matrix A is constant, and one can simply take K to be the 2-norm of A.

Theorem: Let $A(t) = [a_{ij}(t)]$ be a continuous $n \times n$ matrix-valued function on an interval $(a, b) \subseteq \mathbb{R}$. Suppose that $\mathbf{Y}(t)$ and $\widehat{\mathbf{Y}}(t)$ both satisfy the (system of) differential equation(s)

$$\frac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)$$

on the interval, and suppose that $\mathbf{Y}(t_0) = \widehat{\mathbf{Y}}(t_0)$ at some point $t_0 \in (a, b)$. Then $\mathbf{Y}(t) \equiv \widehat{\mathbf{Y}}(t)$ throughout (a, b).

Proof. Any (other) point $t \in (a, b)$ will be contained in some closed interval $[c, d] \subseteq (a, b)$ that contains both t and t_0. Let $\mathbf{Z}(t) \equiv \mathbf{Y}(t) - \widehat{\mathbf{Y}}(t)$; this function also satisfies the same differential equation, and obviously $\mathbf{Z}(t_0) = \mathbf{0}$. By the second Lemma above, $\mathbf{Z}(\cdot) \equiv \mathbf{0}$ throughout $[c, d]$, so in particular $\mathbf{Z}(t) = \mathbf{0}$ and thus $\mathbf{Y}(t) = \widehat{\mathbf{Y}}(t)$—but $t \in (a, b)$ was arbitrary, so the two functions are equal throughout (a, b).

Corollary: The (abstract) vector space consisting of all the $\mathbb{R}^n$-valued functions $\mathbf{Y}(t)$ that satisfy the (system of) differential equation(s) $\dfrac{d\mathbf{Y}(t)}{dt} = A(t)\mathbf{Y}(t)$ on (a, b) is finite-dimensional, and its dimension is the same as the number n of equations of the system.

Proof. Choose a point $t_0 \in (a, b)$. For each of the standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$, use the existence theorem to produce a solution $\mathbf{E}_i(t)$ of the equation $\dfrac{d\mathbf{E}_i(t)}{dt} = A(t)\mathbf{E}_i(t)$ on (a, b) with the property that $\mathbf{E}_i(t_0) = \mathbf{e}_i$, doing this for each $i = 1, \ldots, n$. These solutions are linearly independent, because if one could write

$$c_1\,\mathbf{E}_1(t) + \cdots + c_n\,\mathbf{E}_n(t) \equiv \mathbf{0}$$

then evaluating this equation at $t = t_0$ would produce

$$c_1\,\mathbf{e}_1 + \cdots + c_n\,\mathbf{e}_n = \mathbf{0}$$

and since the l. h. s. of this is the vector $[c_1, \ldots, c_n]^T$ and the r. h. s. is $\mathbf{0}$, we would have all the coefficients $c_1 = \cdots = c_n = 0$. These solutions are also a spanning set: given a solution $\mathbf{Y}(t)$, evaluate it at t_0 and write its value at t_0 in terms of the standard basis:

$$\mathbf{Y}(t_0) = [c_1, \ldots, c_n]^T = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n\ .$$

Then form the solution $\widehat{\mathbf{Y}}(t) = c_1\mathbf{E}_1 + \cdots + c_n\mathbf{E}_n$. This takes the same value at $t = t_0$ that $\mathbf{Y}(t)$ did; therefore, by the uniqueness theorem, it takes the same values that $\mathbf{Y}(t)$ does for all $t \in (a, b)$, which is to say,

$$\mathbf{Y}(t) \equiv \widehat{\mathbf{Y}}(t) = c_1\mathbf{E}_1(t) + \cdots + c_n\mathbf{E}_n(t)$$

and we have exhibited $\mathbf{Y}(t)$ as a linear combination of the $\{\mathbf{E}_i(t)\}_{i=1}^n$. So those solutions form a **basis** of the vector space of all solutions of the (system of) equation(s) $\mathbf{Y}' = A\mathbf{Y}$.

Example: In the important case in which the matrix-of-coefficients function $A(t)$ is a constant matrix A, the solutions of $\mathbf{Y}' = A\mathbf{Y}$ will be defined for all $t \in \mathbb{R}$. Moreover, it is possible to find many solutions of such a system—in many cases, sufficiently many different solutions to form a basis for the solution space—by using the eigenvalues and eigenvectors of the matrix A. The method is illustrated by considering the 2×2 system whose matrix is the one involved in Fraleigh & Beauregard's problem 26, p. 262:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}\ .$$

We need to find numbers λ and vectors $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^2$ for which $A\mathbf{v} = \lambda\mathbf{v}$; since that equation is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$, such an $\mathbf{v}$ can exist if and only if $A - \lambda I$ is singular, which happens if and only if

$$\det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 2) - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$$
$$\lambda = 4 \quad \text{or} \quad \lambda = -1\ .$$

For $\lambda = 4$ we get $A - 4I = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$ and a choice of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ would work (the solution is only determined up to multiplication by a constant); for $\lambda = -1$ we get $A + I = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Whenever one has an eigenvector $\mathbf{w}$ belonging to the eigenvalue λ for the coefficient matrix A of a **constant-coefficient** system $\mathbf{Y}' = A\mathbf{Y}$, the vector-valued function $e^{\lambda t}\mathbf{w}$ will be a solution of the system:

$$\begin{aligned} \frac{d}{dt}(e^{\lambda t}\mathbf{w}) &= \lambda e^{\lambda t}\mathbf{w} = e^{\lambda t}(\lambda \mathbf{w}) \\ &= e^{\lambda t} A\mathbf{w} = A(e^{\lambda t}\mathbf{w}) \end{aligned}$$

so $\mathbf{Y}(t) = e^{\lambda t}\mathbf{w}$ satisfies $\mathbf{Y}' = A\mathbf{Y}$. Applying these general considerations to our system, we find that the two functions $\mathbf{Y}_1(t) = e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{Y}_2(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are solutions of the system. Since we know that the (abstract) vector space of solutions of this system has dimension two and these solutions are obviously not proportional, we see that they are a basis of the space of solutions. For $t_0 = 0$ this is not the basis whose values at t_0 give the standard basis vectors of $\mathbb{R}^2$; the reader might find it interesting to find out how those solutions, which are the functions

$$\mathbf{E}_1(t) = \frac{1}{5}\begin{bmatrix} 2e^{4t} + 3e^{-t} \\ 3e^{4t} - 3e^{-t} \end{bmatrix} \quad \text{and} \quad \mathbf{E}_2(t) = \frac{1}{5}\begin{bmatrix} 2e^{4t} - 2e^{-t} \\ 3e^{4t} + 2e^{-t} \end{bmatrix}$$

respectively, are related to the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ whose columns are the eigenvectors of A.

Linear homogeneous differential equations of higher order can be analyzed using what we have just found out about systems. As an example, consider the equation (Fraleigh & Beauregard's problem 45a, p. 204):

$$y^{(3)} - 9y' = 0 \,.$$

Given a solution $y(t)$ of this equation, form the vector-valued function (with values in $\mathbb{R}^3$) $\mathbf{Y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$.

It is purely a matter of definition to see that

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 9 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix} \tag{\#}$$

so $\mathbf{Y}(t)$ satisfies $\mathbf{Y}' = A\mathbf{Y}$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 9 & 0 \end{bmatrix}$. On the other hand, if $\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$ is a solution of this system and we denote its first-coördinate function by $y(t)$, then the system (#) says about this function $y(t)$ that the second coördinate is $y'(t)$ and the third coördinate is $y''(t)$, and that its derivative $y^{(3)}(t) = 9y'(t)$. The correspondence

$$y(t) \longleftrightarrow \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$$

is a 1-1 correspondence between solutions of $y^{(3)} - 9y' = 0$ and solutions of $\mathbf{Y}' = A\mathbf{Y}$, and it is easy to see that it preserves the vector-space operations. It follows that the uniqueness theorem carries over to the solutions of $y^{(3)} - 9y' = 0$: two solutions of this equation which at some t_0 give the same values of $y(t_0)$, $y'(t_0)$ and $y''(t_0)$ must be the same identical function. The vector space of solutions must again have dimension 3. Finding three linearly independent solutions of $\mathbf{Y}' = A\mathbf{Y}$ by finding eigenvectors of A will yield the same result as the "method of undetermined exponents" in which one tries to find an r for which $y(t) = e^{rt}$ satisfies $y^{(3)} - 9y' = 0$: we have

$$\det \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 9 & 0-\lambda \end{bmatrix} = (-\lambda)(\lambda-3)(\lambda+3)$$

with roots $\lambda = 0,\ 3,\ -3$ and corresponding eigenvectors $[1,0,0]^T$, $[1,3,9]^T$ and $[1,-3,9]^T$ respectively; the corresponding solutions of $\mathbf{Y}' = A\mathbf{Y}$ are then the constant vector $[1,0,0]^T$ and the vector-valued functions $e^{3t}[1,3,9]^T$ and $e^{-3t}[1,-3,9]^T$ respectively, whose first coördinates are the three linearly independent solutions $y(t) = 1$, $y(t) = e^{3t}$ and $y(t) = e^{-3t}$ that would have been obtained by the method

$$\begin{aligned} \text{In}\quad y^{(3)} - 9y' &= 0 \quad \text{let } y = e^{rt} \\ r^3 e^{rt} - 9re^{rt} &= 0 \\ e^{rt}\left(r^3 - 9r\right) &= 0 \quad \text{with roots } r = 0,\ 3,\ -3 \\ y(t) &= 1,\ e^{3t},\ e^{-3t} \end{aligned}$$

that one learns in elementary differential-equations courses. (Of course, one *uses* the simpler method in simple cases like this one, where one is only interested in finding a basis for the vector space of solutions: going through the eigenvalue/eigenvector analysis is much more work than necessary to get that result. The point of replacing the single higher-order equation by a first-order system is that theorem-proving [like our uniqueness theorem above], dimension-counting and analysis of asymptotic behavior [phase-plane analysis, etc.] are easier for the system.)