Math 152 Problems, Week 10

1. In the following series x is a real number. In each case use the ratio test to determine all values of x for which the series converges absolutely.

$$\sum_{n=1}^{\infty} nx^{2n} \qquad \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \sum_{n=1}^{\infty} \frac{x^n}{(n!)^2} \qquad \sum_{n=1}^{\infty} n!x^n$$

2. a) Use the formula

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots \qquad \text{(valid for } |r| < 1\text{)}$$

for the sum of a geometric series to express each of the following functions as a power series $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Give a formula for the coefficient a_n in each case.

$$f(x) = \frac{x}{1-x} \qquad g(x) = \frac{2}{3x+4} \qquad h(x) = \frac{1}{x^4+16}$$

b) For each series in part a), determine the interval of x values in which it converges. Describe what happens at the end points of the convergence interval.

3. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(n+1)!} = 1 - \frac{1}{1!2!}x^2 + \frac{1}{2!3!}x^4 - \frac{1}{3!4!}x^6 + \cdots$$

(remember that $0! = 1$).

a) Use the ratio test to show that the power series converges for all values of x.

b) If N is an integer, we write $s_N(x)$ for the partial sum of the terms of this series up to and including x^N. For example, $s_0(x) = 1$, $s_1(x) = 1$ (there isn't any term with x) and $s_2(x) = 1 - x^2/2$. Plot $s_2(x)$, $s_4(x)$ and $s_6(x)$ on the same axes in the window $-3 \leq x \leq 3$ and $-.5 \leq y \leq 1.5$. (*Calculator Hint:* Key in $s_4(x)$ by adding one term to $s_2(x)$ and likewise key in $s_6(x)$ by adding one term to $s_4(x)$.)

c) Notice in the plots in b) that the three curves are quite different when $|x| > 1$, but are quite close when $|x| \leq 1$. Change the window to $-1 \leq x \leq 1$ and $.5 \leq y \leq 1.5$ to verify this property. Sketch the curves in this new window.

d) Suppose you use the partial sum $s_6(x)$ to approximate the infinite series over the range $-.5 \leq x \leq .5$. Use the alternating series error formula to obtain an upper bound for the error in your approximation. Your answer should be a single number that applies to all x values in the range $-.5 \leq x \leq .5$.

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4. Suppose $f(x)$ is given by a power series of the form

$$a + bx + a(5x)^2 + bx^3 + a(5x)^4 + bx^5 + \cdots + bx^{2k} + a(5x)^{2k+1} + \cdots$$

where a and b are nonzero constants. Find the interval of convergence of the series and a formula for $f(x)$ as a *rational* function. (*Hint:* Write the series as the sum of two series that you know.)