Math 152 Problems, Week 9

Remarks on Calculators and Series: Graphing calculators such as the TI-82 and TI-85 have several built-in functions that make it easy to generate a list of terms in a series and to evaluate partial sums of series. The workshop sheet for week 8 described these functions briefly and had some practice exercises. Refer to that sheet and the instruction manual for your calculator to refresh your memory if necessary.

In this workshop you will analyze two aspects of an infinite series $\sum_{n=0}^{\infty} a_n$:

(1) Determine if the series converges.

(2) If the series converges, find a good numerical approximation to $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$

For answering question (1) first check whether $\lim_{n\to\infty} a_n = 0$. If the answer is 'NO', then the series diverges. If the answer is 'YES', you will need to use one of the 'convergence tests', such as the integral test, the comparison test, or the alternating series test. After you know a series converges, you can use your calculator and the information from the convergence tests to answer question (2)

1. a) Use the comparison test to determine which of the following series converge and which diverge.

$$\sum_{n=1}^{\infty} \ln\left(\frac{5n}{2n+1}\right) \qquad \sum_{n=1}^{\infty} \frac{5+\sin(n)}{1+n^3} \qquad \sum_{n=1}^{\infty} \frac{5+\sin(n)}{1+3^n}$$

b) For each of the convergent series in part a), calculate the sum s_{20} of the first 20 terms. Then estimate the 'tail' of the series $\sum_{n=21}^{\infty} a_n$ by comparing it with an appropriate improper integral or geometric series.

2. For each of the following series, determine whether the three conditions of the alternating series test are satisfied. If the answer is 'YES', calculate the sum of the terms up to $n = 20$ and estimate the 'tail' of the series $\sum_{n=21}^{\infty} a_n$.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^3+4)^2}$ b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n+\ln n}$ c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

3. Use the remainder estimate for an alternating series to answer the following:

a) We will show soon that $\ln 2 = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. How many terms in this series do you need to add in order to get an approximate value for $\ln 2$ with an error less than 0.05?

b) We will show soon that $\dfrac{1}{e} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. If you add up the same number of terms of this series as you did in part a), how close will you be to the true value?

OVER

4. Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ be the nth partial sum of the harmonic series. Then $\lim_{n\to\infty} s_n = \infty$, since this series diverges. In fact, the integral test shows that s_n is about the same size as $\ln n$. Here is a more precise comparison:

a) Define $t_n = s_n - \ln n$. Calculate t_n for $n = 10, 50$ and 100. Fill in the correct inequalities relating these numbers:

$$t_{10} \qquad\qquad t_{50} \qquad\qquad t_{100}$$

b) Show that $t_n > 0$ for *all* integers n. (*Hint:* Compare the area under the graph of $y = 1/x$ and a suitable bar graph representing s_n.)

c) Show that

$$t_n - t_{n+1} = \int_n^{n+1} \frac{dx}{x} - \frac{1}{n+1}.$$

Now draw a picture to show that the right side is positive (it is the difference of two areas). This proves that the sequence $\{t_n\}$ is *decreasing*.

From parts b) and c) and the basic theorem on sequences we see that the limit $\gamma = \lim_{n\to\infty} t_n$ exists. It is called *Euler's constant*, and it is a fundamental constant in mathematics (just like the numbers π and e). Its value (to 15 digits) is

$$\gamma = 0.57721\,56649\,01533\ldots$$