Math 152 Problems, Week 7

Calculators and Sequences: In mathematics, the word *sequence* usually refers to a function $f(n)$ of an *integer variable* n, and a common notation is $f(n) = a_n$. You can graph the pairs $(n, f(n))$ by using *parametric mode* and *dot* graphs. First adjust the settings on your calculator as follows*:

TI-82: Go to `MODE` and select `Par` and `Dot`. Go to `Y=` and set $\texttt{X}_{1T} = \texttt{T}$ and $\texttt{Y}_{1T} = f(\texttt{T})$ (the parameter `T` will play the role of the integer variable n). Then go to `WINDOW` and set $\texttt{T}_{\min}$, $\texttt{T}_{\max}$ and $\texttt{T}_{\text{step}}$ to the appropriate values for n. Set the `X` range the same as the `T` range, and adjust the `Y` range as needed.

TI-85: Go to `MODE` and select `Par`. Go to `GRAPH MORE FORMAT` and select `Dot`. Define $\texttt{X}_{1T} = t$ and $\texttt{Y}_{1T} = f(t)$ (get the parameter t from the `F1` key; it plays the role of the integer variable n). Now set the `WINDOW` as for the TI-82.

1. With your calculator set as described above, graph the following sequences

$$a_n = \frac{3n}{n+5} \qquad b_n = \frac{5n}{n^2+3} \qquad c_n = \frac{n^2}{n+7}$$

for $n = 1, 2, 3, \ldots, 10$. Compute the values of each sequence for $n = 1$ and $n = 2$ by hand to be sure that your calculator is set up correctly. (Remember to key in the variable n as T or t.) These sequences have obvious limits as $n \to \infty$. (What are they?) Check that these limits are suggested (although not proved) by the graphs.

2. Consider the sequences

$$a_n = \left(1 + \frac{1}{n}\right)^n \qquad b_n = \left(1 + \frac{1}{n^2}\right)^n \qquad c_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

a) Use your calculator to plot the first ten terms of each of these sequences. Try to guess $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ and $\lim\limits_{n\to\infty} c_n$ from your calculator information.

b) Treat n as a continuous variable and use l'Hospital's rule to prove your guesses in a).

Note This is a *very* important example. All of these sequences "sort of" look like $(1+0)^\infty$ in a very loose way (when $n \approx \infty$), but each behaves differently as n gets large.

Factorials will be important in many of the applications we'll study. If you need $n! = n(n-1)\cdots 2\cdot 1$ in a formula, you get ! from the `MATH` and `PRB` menu on the TI-82. On the TI-85, you get ! from the `CHAR` menu.

3. a) Let $a_n = \dfrac{5^n}{n!}$. Graph the sequence $\{a_n\}$ for $n = 0, 1, \ldots, 5$ and guess $\lim_{n\to\infty} a_n$. Then graph the sequence for $n = 0, 1, \ldots, 20$ and guess $\lim_{n\to\infty} a_n$. Comment?

b) Let $b_n = \dfrac{a_{n+1}}{a_n}$, with a_n as above. Find an algebraic formula for b_n, and then graph b_n. Find $\lim_{n\to\infty} b_n$.

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* Check the manuals for more details. Other graphing calculators have equivalent capabilities.

4. Define $a_1 = 2$ and $a_n = \sqrt{3 + a_{n-1}}$ for $n > 1$. Sequences of this type are called "recursively defined": here the second term is $\sqrt{3 + \text{the first term}} = \sqrt{3+2} = \sqrt{5}$ and the third term is $\sqrt{3 + \text{the second term}} = \sqrt{3 + \sqrt{5}}$, etc. Recursively defined sequences occur in many applications.

a) Compute a_n for $n = 2, 3, \ldots, 10$. You can use the `ANS` key to calculate a_{n+1} from a_n. Record the values of a_n and plot them (by hand) on a graph. Does it appear that $\lim_{n\to\infty} a_n$ exists?

In the next two parts of this problem you will show that the sequence $\{a_n\}$ satisfies the hypotheses of the following **Important Theorem:**

If $\{a_n\}$ is an *increasing* and *bounded* sequence, then $\lim_{n\to\infty} a_n$ exists.

b) Verify that the sequence $\{a_n\}$ is increasing. Do this by showing two things:

i) $0 \le a_1 \le a_2$.   ii) If $0 \le A \le B$ then $0 \le \sqrt{3+A} \le \sqrt{3+B}$.

Then use these **two** facts and the method of mathematical induction to prove that the following **infinite** string of inequalities is true:

$$a_1 \le a_2 \le a_3 \le a_4 \le \ldots \le a_n \le a_{n+1} \le a_{n+2} \le \ldots$$

Note: This isn't silly. For example, is $a_{345,678} \le a_{345,679}$? You certainly don't want to compute all these numbers exactly!

c) Now verify that every number in the sequence $\{a_n\}$ is less than 3. Do this by showing two things:

i) $0 \le a_1 < 3$.   ii) If $0 \le A < 3$, then $0 \le \sqrt{3+A} < 3$.

Then use these **two** facts and the method of mathematical induction to prove that $a_n < 3$ for all n.

Note: Again, the conclusion of this section of the problem is not at all obvious. You don't want to compute $a_{123,456,789}$, and yet you have now proved that it is less than 3 for sure.

d) From b) and c) (and the **Important Theorem**) we now know that $L = \lim\limits_{n\to\infty} a_n$ exits. To get an equation for L take the limit of the equation $a_n = \sqrt{3 + a_{n-1}}$ as $n \to \infty$ and do a little algebra. Now solve this equation for L and compare your solution to the numbers gotten in a).