Math 152 Problems, Week 5

1. a) Consider the following improper integrals:

$$\int_1^\infty e^{-x^2}\,dx \qquad\qquad \int_1^\infty xe^{-x^2}\,dx$$

Graph the integrands and determine which integral is larger. Do the integrals converge or diverge? (*Hint:* Try a substitution to evaluate one of the integrals.)

b) Consider the following improper integrals:

$$\int_1^\infty \frac{\ln x}{x}\,dx \qquad\qquad \int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$$

Graph the integrands and determine which integral is larger. Do the integrals converge or diverge? (*Hint:* Try a substitution to evaluate one of the integrals.)

Note: The following problems on numerical integration refer to the programs `TRAP` and `SIMP` that you should have already keyed into your calculator (these programs are described on a separate handout).

2. a) Calculate an approximation to $\int_0^4 x^3\,dx$ using the Trapezoidal Rule with $n = 4$. Write out the calculations by hand, then check using the program TRAP (remember to set $y_1 = x^3$ in your calculator before running the program).

b) Sketch the graph of $y = x^3$ for $0 \le x \le 4$. In your sketch show the area that the trapezoidal rule calculates exactly. Compare your approximation from a) with the exact value of the integral obtained by the Fundamental Theorem of Calculus. Use the error formula for the Trapezoidal Rule (text, boxed formula on p. 460) to predict the error in this approximation. How could you have predicted from your graph that the approximation would be larger than the true value of the integral?

c) Calculate an approximation to $\int_0^4 x^3\,dx$ using Simpson's Rule with $N = 4$. Write out the calculations by hand, then check using the program SIMP. Compare your approximation with the exact value. Before beginning the calculation you could have predicted from the error formula for Simpson's rule (text, boxed formula on page 464) that the error would be as astoundingly small as it turned out to be. Why?

d) Suppose you use Simpson's rule with $N = 8$ to approximate $\int_0^4 x^4\,dx$. Use the error formula for Simpson's rule (text, boxed formula on page 464) to predict the error in this approximation. Now use the program SIMP to obtain this approximation and find the actual error in the approximation.

(Over)

3. a) Let $a \geq 1$. Use a substitution to calculate $\int_a^\infty x e^{-x^2}\, dx$. Now use this result and the method of Problem # 1 to show that $\int_a^\infty e^{-x^2}\, dx \leq \frac{1}{2} e^{-a^2}$.

b) Calculate an approximate value for the convergent improper integral $\int_0^\infty e^{-x^2}\, dx$ as follows: Find a value of a so that $\frac{1}{2}e^{-a^2} < 10^{-5}$. Now calculate $\int_0^a e^{-x^2}\, dx$ approximately using the `fnInt(` program on your calculator. Compare your calculator value with the *exact value*

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Explain why the two numbers should agree to four decimal places. (You will meet this integral in probability and statistics)

c) Use the exact numerical value of the integral in part b) and integration by parts to calculate $\int_0^\infty x^2 e^{-x^2}\, dx$. (*Hint:* Write the integrand as $x(xe^{-x^2})$.)

4. Consider the definite integral $\int_0^1 \cos(x^2)\, dx$. Read all parts of the problem before doing part a), so as to save recaculating the needed numbers (you can use the `ANS` key on your calculator to do calculations on the output of the numerical integration programs.)

a) Compute the results of approximating this integral with both the Trapezoidal Rule and Simpson's Rule with $n = 8$ and $n = 16$ and $n = 32$ subdivisions. (If you have already keyed the `TRAP` and `SIMP` programs into your graphing calculator, this will take less than two minutes to get the six approximations; otherwise, it's impossibly time-consuming.) Before making the calculations, set your calculator to 9 digits of accuracy and `Sci` using the `MODE` key.

b) A twenty-digit approximation of the true value of this integral is

$$.90452423790027208145$$

Compute the differences between each of the approximations obtained in a) and the true value. (It will save time if you store this value in your calculator using the `STO` key BEFORE doing the calculations in a).) Then multiply the Trapezoidal Rule errors by n^2 and the Simpson's Rule errors by n^4 in each case (reason: the theoretical error formulas for the Trapezoidal Rule and Simpson's Rule). Now make a table exhibiting these numbers and use the information in the table to answer the following questions:

i) When n is doubled, what seems to happens to the error (for each Rule)?

ii) Predict (without calculation) what the error would be when $n = 64$, based on the cases $n = 8, 16, 32$. Then check your prediction by doing the calculation (for each Rule).