Calculus 152 Problems, Week 2

1. a) Sketch the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

b) Find the area of the region in a) in *two ways*: first use x as the variable of integration; then use y as the variable of integration. Be careful in setting up the limits on the integrals.

2. Here are two different ways to approximate the function $\sin(x)$ on the interval $0 \le x \le \pi$.
a) Let $f(x) = cx(\pi - x)$, where c is a constant. Note that $f(0) = 0$ and $f(\pi) = 0$ for any choice of c. Now find the value of c so that

$$\int_0^\pi f(x)\,dx = \int_0^\pi \sin(x)\,dx$$

(thus the areas under the curves $y = f(x)$ and $y = \sin(x)$ for $0 \le x \le \pi$ are *exactly* the same, and the two curves have the same endpoints).

b) Let $g(x) = bx(\pi - x)$, where b is a constant. Note that $g(0) = 0$ and $g(\pi) = 0$ for any choice of b. Determine b so that the the curves $y = g(x)$ and $y = \sin(x)$ have the same tangent lines at $(0, 0)$ and at $(\pi, 0)$.

c) Plot the graphs of $y = \sin(x)$, $y = f(x)$, and $y = g(x)$ for $0 \le x \le \pi$ on the same axes.

3. Let R be the parabolic region in the x-y plane bounded below by the curve $y = x^2$ and above by the line $y = 1$.
a) Sketch R. Set up and evaluate an integral that gives the area of R.

b) Suppose a solid has base R and the cross-sections of the solid perpendicular to the y-axis are squares. Sketch the solid and find its volume.

c) Suppose a solid has base R and the cross-sections of the solid perpendicular to the y-axis are equilateral triangles. Sketch the solid and find its volume.

4. The *Bernoulli polynomials* $B_n(x)$ are defined by $B_0(x) = 1$ and

$$B_n'(x) = B_{n-1}(x), \qquad \int_0^1 B_n(x)\,dx = 0$$

for $n = 1, 2, 3, \ldots$. For example, $B_1'(x) = B_0(x) = 1$, so $B_1(x) = x + C$ for some constant C. The condition

$$0 = \int_0^1 (x + C)\,dx = [x^2/2 + Cx]_0^1 = 1/2 + C$$

then forces $C = -1/2$. Hence $B_1(x) = x - 1/2$.
a) Find formulas for $B_2(x)$ and $B_3(x)$ and graph these polynomials on the interval $0 \le x \le 1$.

b) Use the fundamental theorem of calculus to show that $B_n(0) = B_n(1)$ for all values of $n \ge 2$.