Problems, Week 3

1. a) There is a rational number*, A, so that

$$\int_2^3 \frac{1}{x(x+1)} dx = \ln A \ .$$

What is A?

b) There is an algebraic number**, B, so that

$$\int_3^4 \frac{1}{x(x+1)(x+2)}\, dx = \ln B$$

What is B?

2. A program called *Macsyma* can do integration. It is a bit old, but it can do 4 of the following 5 integrals in a total of less than 5 seconds of CPU time. Please do 3 of these integrals. (You choose which ones!)

a) $\displaystyle\int x \sin(x^2)\, dx$

b) $\displaystyle\int x\,(\sin x)^2\ dx$

c) $\displaystyle\int x^2 \sin x\, dx$

d) $\displaystyle\int x^2\,(\sin x)^2\ dx$

e) $\displaystyle\int x^2 \sin(x^2)\, dx$

3. <u>A lesson in approximating $\sin x$</u>

a) For x near 0, $\sin x$ is well-approximated by its tangent line at $x = 0$. What is this tangent line?

b) For many purposes, approximation over an interval is preferred over approximation near a point. There are many criteria for assessing the accuracy of such approximations. One is *mean-square error.* The mean-square error between a straight line $y = Ax$ going through the origin and the function $\sin x$ over the interval $[0, 1]$ is given by the definite integral

$$\int_0^1 (\sin x - Ax)^2\ dx$$

Find the A which <u>minimizes</u> this integral. (Hint: expand the integrand and compute everything. You'll get a simple function of A. Minimize this function.)

c) Construct a table, the first column of which is x, as x ranges from 0 to 1 in increments of $\frac{1}{10}$. The other three columns of the table should be the corresponding values of $\sin x$, of the

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* A rational number is a quotient of two integers

** An algebraic number is a number which can be obtained from integers using addition, subtraction, multiplication, division <u>and</u> root extractions.

tangent line approximation to $\sin x$ obtained in a), and of the mean-square approximation, Ax, to $\sin x$ obtained in b). *Use a calculator!*

d) Make a rough sketch over the unit interval $[0,1]$ of the three curves whose data points you have tabulated in c).

4. a) When D is very large, show that the integral $\int_1^2 e^{-Dx}\,dx$ is very small. (You may wish to draw a picture, but other verification is also necessary.)

b) When D is very large, show that the integral $\int_1^2 xe^{-Dx}\,dx$ is very small. (You may wish to draw a picture, but other verification is also necessary.)

c) When D is very large, show that the integral $\int_1^2 \frac{1}{1+5x^{48}}e^{-Dx}\,dx$ is very small†. (You may wish to draw a picture, but other verification is also necessary.)

5. Consider the function $G(x) = e^x \sin Nx$ <u>on the interval</u> $[0,1]$.

a) With a sketch or otherwise, describe this function when $N = 3$ and when $N = 10$ and when $N = 10^{10}$.

b) Compute $\int_0^1 G(x)\,dx$ if $N = 10^{10}$.

c) Explain b)'s result in relation to a).

6. a) Compute $\int_1^2 \frac{dx}{x^2}$.

b) Compute $\int_1^2 \frac{dx}{x(x-m)}$ if m is a small positive number.

c) What happens to your answer to b) when $m \to 0^+$?

d) Compute $\int_1^2 \frac{1}{x^2+n}\,dx$ if n is a small positive number.

e) What happens to your answer to d) when $n \to 0^+$?

f) Pictures? Comments??

7. Many integrals can be done with "rationalizing substitutions" which change the integrals into integrals involving rational functions. In turn, these integrals can be computed (theoretically) using partial fractions. Here are two examples of such integrals:

† It is **not** always necessary or even possible to compute every integral exactly. But if the integral can be estimated easily . . .

a) Integrate:

$$\int \frac{1}{(\sqrt{x}+1)(\sqrt{x}+2)}\,dx$$

Try $x = t^2$.

b) Integrate:

$$\int \frac{e^x+1}{e^{2x}+1}\,dx$$

Try $t = e^x$.

8. The following information is known about a function T and its derivatives:

x	$T(x)$	$T'(x)$	$T''(x)$
1	2	-2	2
2	3	6	5
3	7	4	-4
4	2	5	7

a) Compute $\displaystyle\int_2^3 T'(x)\,dx$.

b) Compute $\displaystyle\int_2^3 T''(x)\,dx$.

c) Compute $\displaystyle\int_2^3 x dx$.

d) Compute $\displaystyle\int_2^3 xT''(x)\,dx$.

Better not look at b) and c)! Integrate by parts.

e) Compute $\displaystyle\int_2^3 x^2T'''(x)\,dx$.

And again and again.