Calculus 151 Problems, Week 8

1. a) Sketch a graph of $F(x) = e^x$, find its derivative, give an equation for the line tangent to the curve at $x = 2$, and sketch on the same graph the line tangent to the curve when $x = 2$. Note any horizontal or vertical asymptotes on the graph.

The exponential function: the world's friendliest function.

b) Sketch a graph of $G(x) = e^{-\frac{1}{x}}$, find its derivative, give an equation for the line tangent to the curve at $x = 2$, and sketch on the same graph the line tangent to the curve when $x = 2$. Note any horizontal or vertical asymptotes on the graph.

The exponential function: the world's weirdest function.

2. Consider the functions $y_1 = (1 + x/5)^5$, $y_2 = (1 + x/10)^{10}$, and $y_3 = e^x$.

(a) Graph y_1, y_2, and y_3 in the following windows (all three graphs in each window):

$$\{-1 \leq x \leq 1,\, 0 \leq y \leq 3\}, \quad \{-2 \leq x \leq 2,\, 0 \leq y \leq 6\}, \quad \{-3 \leq x \leq 3,\, 0 \leq y \leq 20\}.$$

Is it hard to distiguish the graphs in the first window? What about the third window?

(b) By calculator experimentation, find an interval $-A \leq x \leq B$ (where A and B are positive numbers) so that

$$|(1 + x/5)^5 - e^x| < 0.1$$

for all x in this interval.

(c) By calculator experimentation, find an interval $-C \leq x \leq D$ (where C and D are positive numbers) so that

$$|(1 + x/10)^{10} - e^x| < 0.1$$

for all x in this interval. This interval should be bigger than the interval you found in (b).

(d) Recall that $e = \lim_{n\to\infty}(1 + 1/n)^n$. Use this to prove that if x is any fixed number, then

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

How is this result related to (a), (b), and (c)?

3. Although the function $f(x) = x^{10} + 1$ grows more rapidly than $g(x) = e^x$ when x is small, eventually $g(x)$ gets 'infinitely bigger' than $f(x)$ as $x \to +\infty$ (exponentials always beat powers at infinity). Here we explore this property using the function $h(x) = g(x)/f(x)$

(a) Graph $h(x)$ on the interval $0 \leq x \leq 5$. Based on this graph, what is your guess for $\lim_{x\to+\infty} h(x)$?

(b) Now graph $h(x)$ on the interval $0 \leq x \leq 30$. Does the result confirm your guess in (a)?

(c) Now graph $h(x)$ on the interval $0 \leq x \leq 40$. Do you want to revise your guess?

(Over)

4. Consider the function

$$H(x) = \frac{e^{Ax}}{1 + e^{Ax}}$$

where A is a constant. The graph of $H(x)$ in a fixed interval, say $-1 \leq x \leq 1$, can have many different shapes, depending on the size of A. Try the following examples (if you key the function into your calculator using the standard variable X and the alpha variable A, then you can set the value of A by entering a number and pressing `STO` A):

(a) Graph $y = H(x)$ for x in $[-1, 1]$ if $A = 1$.

(b) Graph $y = H(x)$ for x in $[-1, 1]$ if $A = 10$.

(c) Graph $y = H(x)$ for x in $[-1, 1]$ if $A = 100$. Now explain how to obtain this graph (approximately) WITHOUT using a calculator.

(d) Graph $y = H(x)$ for x in $[-1, 1]$ if $A = 1/10$.

(e) Graph $y = H(x)$ for x in $[-1, 1]$ if $A = 1/100$. Now explain how to obtain this graph (approximately) WITHOUT using a calculator.

5. Define the functions $f_n(x) = n(x^{1/n} - 1)$ for $n = 1, 2, 3, ...$ and $x \geq 0$.

(a) Graph $f_2(x)$, $f_4(x)$, $f_8(x)$ and $f_{16}(x)$ for $0 \leq x \leq 10$. What features do the graphs have in common? When $x \geq 1$, which function is the biggest? Which function is the smallest? What is $\lim_{n\to\infty} f_n(0)$?

(b) Now graph the function $\ln x$ on the same axis. What limit result does your family of graphs illustrate?