MATH 151/153 FINAL EXAM: December 16, 1998, 4–7 p.m.Name: ______________________

TIME: 180 minutes. INSTRUCTIONS: Do all problems. SHOW ALL YOUR WORK. Full credit may not be given for an answer alone.

You may use a TI-82 calculator or equivalent. You may have one handwritten $8.5'' \times 11''$ **sheet of formulas. You may NOT use a TI-92 or any calculator with a QWERTY keyboard.**

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(10) 1. Write the definition of the derivative $f'(x)$ as a limit and then use this definition to find the derivative of $f(x) = 3x^2 - 2x + 4$. Do not use any rules for differentiation. Be sure to show all your work.

(16) 2. Find the areas of the following regions. Graphing is recommended as a preliminary step. Do not give decimal calculator read-outs, but give exact answers, in terms of well-known constants such as π, e, $\ln 3$, $\sqrt{2}$, etc.

(a) The region above the x-axis, to the left of the line $x = 5$, and below the curve $y = e^x - 1$.

(b) The region above the x-axis, between the lines $x = 1$ and $x = 2$, and below the curve $y = \dfrac{2}{x} + \sin(\pi x)$.

(18) 3. Compute the following limits. Do not give decimal approximations to the answers, but give exact answers in terms of whole numbers and well-known constants such as π, e, $\ln 3$, $\sqrt{2}$, etc.

(a) $\displaystyle\lim_{x \to 0} \frac{\tan 7x}{\sin 13x}$

(b) $\displaystyle\lim_{x \to +\infty} \frac{\sqrt{4 + 5x^4}}{3 + 6x + 7x^2}$

(c) $\displaystyle\lim_{x \to 1} \frac{\ln(2x^2 - 1)}{(x - 1)e^x}$

(10) 4. For the curve $y^3 + 5xy^2 + x^4 = -16$, find

(a) $\dfrac{dy}{dx}$

(b) an equation of the line tangent to the curve at $(-2, 2)$.

(30) 5. Let $f(x) = \dfrac{x^2 - 1}{x - 5}$. (a) Show that $f'(x) = \dfrac{x^2 - 10x + 1}{(x - 5)^2}$.

(In the questions below, you may find the derivative more convenient to use in the equivalent form $f'(x) = 1 - \dfrac{24}{(x - 5)^2}$.)

(b) Use calculus to determine the intervals where $y = f(x)$ is increasing and where it is decreasing, as well as the location of all local maxima and minima. Give exact values, not decimal approximations.

(c) Use calculus to determine the intervals where $y = f(x)$ is concave upwards and where it is concave downwards, and the location of all points of inflection. Again give exact values.

continued. . .

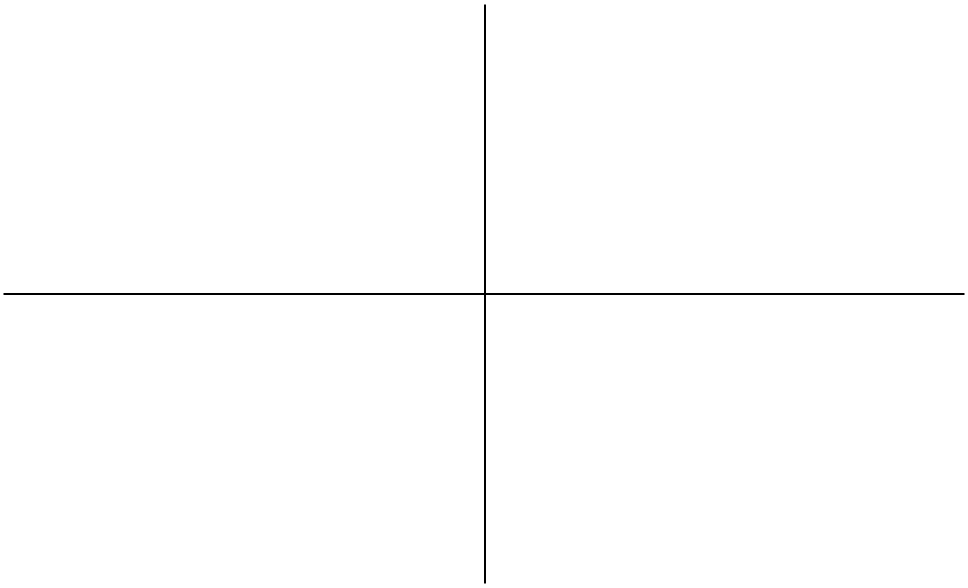
5, continued (recall $f(x) = \dfrac{x^2-1}{x-5}$)

(d) Determine algebraically the intervals where $y = f(x)$ is positive and where it is negative.

(e) Determine the location of any vertical asymptotes of the graph of f.

(f) Determine $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(g) Sketch the graph of $y = f(x)$ on the axes below, labelling all critical and inflection points and showing all horizontal and vertical asymptotes. Then give the specifications of a window in which all the features above are clearly visible.



(30) 6. Find the following. In (c) and (d), give exact answers, not decimal approximations.

(a) $\displaystyle\int \frac{\cos 2x}{1+(\sin 2x)^2}\,dx$

(b) $\displaystyle\int \left(\frac{4}{t^2} + \frac{1}{\sqrt{1-t^2}}\right) dt$

(c) $\displaystyle\int_{-1}^{1} f(x)\,dx$, where $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x^3, & \text{if } x \geq 0. \end{cases}$

(d) $\displaystyle\int_0^{\pi/4} (\sin x + \cos x)^2\,dx$

continued. . .

6, continued

(e) $\displaystyle\frac{d}{dx}\int_3^x \sqrt{t^7+1}\,dt$

(16) 7. A rocket has an initial velocity of 1 km/s, and at $t = 0$ it starts to accelerate. The acceleration at time t is given by the formula $a(t) = 0.06t$ km/s^2.

(a) Find a formula for the velocity of the rocket at all times $t \geq 0$.

(b) Find the distance traveled by the rocket between times $t = 0$ and $t = 4$.

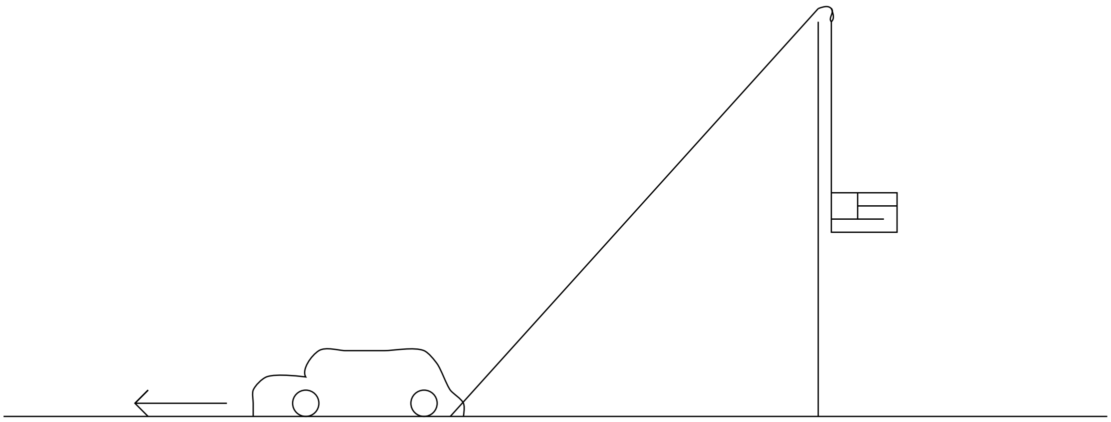
(15) 8. Let $y = f(x)$ be a function whose derivative is $f'(x) = (9 - x^2)^{1/3}$ and such that $f(1) = 5$.

(a) Find the linear approximation $L(x)$ for $f(x)$ around $x = 1$. (Instead, for full credit, you may obtain a formula for the differential dy around $x = 1$.)

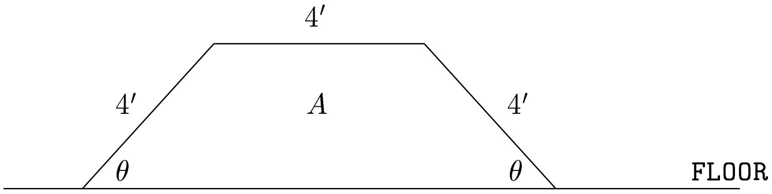
(b) Using your answer to (a), obtain an approximate numerical value for $f(1.1)$.

(c) Determine whether the approximate value which you obtained for $f(1.1)$ is greater or smaller than the true value. Give an *explanation* involving $f''(x)$ and tangent lines.

(15) 9. A flagpole is 40 feet high and stands on level ground. A flag is attached to a 120 foot rope passing through a pulley at the top of the flagpole. The other end of the rope is tied to a car at ground level. If the car is driving directly away from the flagpole at 3 ft/sec, how fast is the flag rising when the top of the flag is 20 feet off the ground?



(15) 10. A child wants to build a tunnel using three equal boards, each 4 feet wide, one for the top and one for each side, as shown. The two sides are to be tilted at equal angles θ to the floor. What is the maximum cross-sectional area A that can be achieved?



(15) 11. The half-life of carbon-14 is 5715 years. A sample of papyrus from a 19th dynasty Egyptian manuscript has *lost* one third of its carbon-14 through radioactive decay.

(a) How old would you say this document is?

(b) In what year will the sample have lost 40% of its original carbon-14?

(10) 12. Suppose that $f(0) = 2$ and $f'(x) = \dfrac{1}{3 + \cos^2 x}$ for all x, $0 \le x \le \dfrac{\pi}{2}$. Using the Mean Value Theorem, you can conclude that

$$A \le f\left(\frac{\pi}{2}\right) \le B$$

for certain numbers A and B. What are A and B?