Math 152, Spring 2009, Review Problems for Exam 2

Your second exam is likely to have problems that do not resemble these review problems.

(1) Solve: (a) $dx/dt = \tan x$, $x(0) = \pi/6$. (b) $(4 + x^3)^{1/2}(dy/dx) = (xy)^2$, $y(0) = -1$.

(2) For each improper integral below, determine convergence or divergence. Evaluate those that converge.

$$\int_0^\infty \frac{x^2\,dx}{e^{2x}}, \quad \int_0^{\pi/2} \sec x\,dx\ , \quad \int_0^1 \ln x\,dx\ , \quad \int_3^\infty \frac{dx}{(x^3 - x)^{1/4}}.$$

(3) Consider the function $f(x) = \ln x$. Find the Taylor polynomial $T_5(x)$ for $f(x)$ centered at $x = 1$. Evaluate $T_5(3/2)$. Find an estimate for $|\ln(3/2) - T_5(3/2)|$ using the Error Bound. Later on, we will see that $\ln(3/2)$ can be written exactly as the alternating sum

$$\ln(3/2) = \frac{(1/2)}{1} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} - \frac{(1/2)^6}{6} + \cdots\ .$$

Find another estimate for $|\ln(3/2) - T_5(3/2)|$ using this alternating sum, your value of $T_5(3/2)$ and the theory of alternating series. Is one estimate better than the other?

(4) This problem involves a very picky person who only accepts coffee made from beans grown on one particular hilltop, known to the locals as the Coffee Mound. In addition, the coffee must have a temperature of 35^oC when this person drinks it.
(a) This individual's favorite restaurant serves that coffee at 50^oC in a room with an ambient temperature of 25^oC. If the coffee cup is not touched, then its contents reach 30^oC one hour later. The picky patron demands that coffee be brought in with the main course, but only drinks it when the dessert arrives. How many minutes should the waiter allow between the main course and dessert to fulfill the 35^oC requirement?
(b) Our coffee story takes place in the 25th century, when positional astronomy is so precise that the following prediction can be made decades in advance: A car-sized asteroid named Decaf will punch a parking-lot-sized impact crater into the Coffee Mound in the year 2439. Our coffee consumer was aware of this in 2409 when he invested in an annuity that would pay out 1,000 dollars per year to finance the purchase of the special coffee. Since this arrangement only made sense as long as the Coffee Mound kept producing, the annuity was designed to run out of money in 2439, when the tabloids will finally publish the long anticipated headline "Decaf decaffeinates Coffee Mound." How many dollars were invested in 2409 if the interest rate was 5 percent?

(5) Write the repeating decimal 0.714714714... as a ratio of two integers.

(6) A student writes $\infty = 1 + 3 + 3^2 + 3^3 + 3^4 + \cdots = \frac{1}{1-3} = -\frac{1}{2}$. Obviously, there is something wrong here. What is it that the student forgot to check?

(7) Find the exact values of the sums $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + (-5)^n}{7^n}\ ,\ \sum_{n=4}^{\infty} \frac{1}{n(n+1)}\ ,\ \sum_{n=4}^{\infty} \frac{1}{n(n+2)}\ .$

MORE PROBLEMS ON THE NEXT PAGE

(8) Evaluate the limits $\lim_{n\to\infty}(3n)^{1/n}$, $\lim_{n\to\infty} n^2\big(1-\cos(1/n)\big)$, $\lim_{n\to\infty}\left(1-\frac{5}{n}\right)^n$. Hints: L'Hôpital's Rule, logarithms and $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

(9) Use appropriate tests to determine convergence or divergence for each of the following series.

$$\sum_{n=7}^{\infty}\frac{(-1)^n(n+4)}{n+3} \qquad \sum_{n=2}^{\infty}\frac{7^n n^7}{8^{n+3}} \qquad \sum_{n=2}^{\infty}\sqrt{\frac{n^2+1}{n^5-n^4-2}} \qquad \sum_{n=3}^{\infty}\frac{\sin^2(e^n)}{n^{4/3}} \qquad \sum_{n=1}^{\infty}\frac{n^n}{n!5^n}$$

$$\sum_{n=3}^{\infty}\frac{3n^2-n-1}{5n^3+n^2+5} \qquad \sum_{n=1}^{\infty}\frac{3^n+n}{4^n-3^n} \qquad \sum_{n=1}^{\infty}\frac{n!n!(1.7)^n}{(2n)!} \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}} \qquad \sum_{n=3}^{\infty}\frac{\cos(n^3)}{n(\ln n)^2}$$

(10) Use the Limit Comparison Test to determine convergence or divergence for each of the following series: $\sum_{n=5}^{\infty}\big(1-\cos(1/n)\big)$, $\sum_{n=5}^{\infty}\Big(1-\cos\big(1/\sqrt{n}\big)\Big)$. Hint: See problem (8).

(11) Euler solved a very difficult problem when he found that $\pi^2/6$ is the exact value of the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Once we know this fact, it is much easier to find the exact value of the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Evaluate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ by computing $\sum_{n=1}^{\infty}\frac{2}{(2n)^2}$ and subtracting this last sum from $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(12) We have $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=S$, where S is a number that you are asked to compute in problem (11). How many terms of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are required to approximate $\pi^2/6$ with an accuracy of 10^{-6}? How many terms of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ are required to approximate S with an accuracy of 10^{-6}? These two series give us two different ways of approximating π^2. Which way is more efficient?

(13) A Math 152 instructor wants to choose a particular nonzero constant c that will make it possible for his students to find the exact length of the curve $y = cx^2 - \ln x$ over the interval $[2, 3]$. What value of c should be chosen by this instructor? What is the length of the curve when the proper c is used? Hint: What condition on a and b will give us $1 + (a - b)^2 = (a + b)^2$?

(14) Find the area of the surface obtained by rotating the curve $y = \sqrt{x+1}$, $1 \le x \le 2$ about the x-axis.