Partial solutions to the final exam review problems

(1) The area is $\displaystyle\int_{\pi/3}^{4\pi/3} \sin x - \sqrt{3}\cos x\, dx = 4.$

(2) We find $\int_1^4 f(x)\,dx = \int_1^{10} f(x)\,dx - \int_4^{10} f(x)\,dx = 2 - 7 = -5$. Similarly, we get $\int_8^{10} f(x)\,dx = \int_1^{10} f(x)\,dx - \int_1^8 f(x)\,dx = 2 - 14 = -12$. Finally, $\int_4^8 f(x)\,dx = \int_1^{10} f(x)\,dx - \int_1^4 f(x)\,dx - \int_8^{10} f(x)\,dx = 2 - (-5) - (-12) = 19$.

(3) The integrals are $2x + \dfrac{5x^2}{2} + x^3 + C$, $2\ln\left(\dfrac{2}{3}\right) - \dfrac{15}{2}$ and $\dfrac{5}{2}$.

(4) The integrals are $\dfrac{e^{x^3+4}}{3} + C$, $\dfrac{\sin^2 x}{2} + C$ and $\dfrac{\tan^2 x}{2} + C$.

(5) It takes $\dfrac{7\ln 3}{\ln 4}$ days.

(6) It is continuous because $\lim_{x\to 1-} 2x+1 = \lim_{x\to 1+} 4x-1 = 3 = f(1)$. If it were differentiable at 1 then the derivative of $2x+1$ at 1 would be the same as the derivative of $4x-1$ at 1, but this would imply $2 = 4$, which is a contradiction.

(7) Continuity at 0 implies $a = \lim_{x\to 0+} f(x) = \lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \dfrac{\ln x}{1/x} = \lim_{x\to 0+} \dfrac{1/x}{-1/x^2} = 0$, where we used l'Hôpital's Rule at the end. If the function were differentiable at 0 then we would have $f'(0) = \lim_{x\to 0+} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \dfrac{x\ln x}{x} = \lim_{x\to 0+} \ln x = -\infty$, and $f'(0)$ would not be a number. We conclude that the function is not differentiable at 0. The function is decreasing on $(0, e^{-1})$ and increasing on (e^{-1}, ∞). By symmetry, it is also increasing on $(-e^{-1}, 0)$ and decreasing on $(-\infty, -e^{-1})$.

(8) The function is concave down on $(0, e^{-3/2})$. It is concave up on $(e^{-3/2}, \infty)$.

(9) The function is increasing on $(-1, 0)$ and $(1, \infty)$. The function is decreasing on $(-\infty, -1)$ and $(0, 1)$. There is a local maximum at $x = 0$ and there are local minima (which happen to be absolute minima) at $x = -1$ and $x = 1$. The function is concave down on $(-\sqrt{3/5}, \sqrt{3/5})$. The function is concave up on $(-\infty, -\sqrt{3/5})$ and $(\sqrt{3/5}, \infty)$. There are inflection points at $x = -\sqrt{3/5}$ and $x = \sqrt{3/5}$.

(10) The four limits are 0, 0, 25/49 and 1/3, respectively.

(11) The function is concave down on $(-(3/4)^{1/4}, (3/4)^{1/4})$. The function is concave up on $(-\infty, -(3/4)^{1/4})$ and $((3/4)^{1/4}, \infty)$.

(12) The absolute maximum occurs at $x = 5/9$.

(13) The horizontal asymptotes are $y = 1$ and $y = -1$. The vertical asymptote is $x = -2$.

(14) The closest points are $\left(-\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$

(15) The six derivatives are, respectively,

$$\frac{2xe^{x^2}}{1+e^{2x^2}} \ , \ \frac{1/x}{\sqrt{1-(\ln x)^2}} \ , \ \frac{3x^2}{|x^3|\sqrt{x^6-1}} = \frac{3}{|x|\sqrt{x^6-1}},$$

$$\frac{1}{(1+x^2)\tan^{-1} x} \ , \ -2\cos(e^{5x}+x^2)\sin(e^{5x}+x^2)(5e^{5x}+2x) \ , \ x^x(1+\ln x) \ .$$

(16) The three second derivatives are, respectively,

$$20(x^2+1)^9 + 360x^2(x^2+1)^8 \ , \ \frac{2x^3-6x}{(x^2+1)^3} \ , \ \frac{x}{(1-x^2)^{3/2}} \ .$$

(17) The tip of the hour hand travels $12(2\pi)$ feet in 12 hours. This means that the tip of the hour hand moves with a speed of 2π feet per hour. The tip of the minute hand travels $16(2\pi)$ feet in 1 hour. This means that the tip of the minute hand moves with a speed of 32π feet per hour. We will use a coordinate plane which has the center of the clock at $(0, 0)$. Let (x, y) be the location (in feet) of the tip of the hour hand. Let (u, v) be the location (in feet) of the tip of the minute hand. Let s be the distance between the tip of the hour hand and the tip of the minute hand. The variables x, y, u, v, s are functions of time t (measured in hours). At 9 pm we have $x = -12$, $y = 0$, $u = 0$, $v = 16$, $s = 20$. What we know about speeds (and other considerations) gives us the following information:

$$\text{At 9 pm, } \frac{dx}{dt} = 0 \ , \ \frac{dy}{dt} = 2\pi \ , \ \frac{du}{dt} = 32\pi \ , \ \frac{dv}{dt} = 0 \ .$$

If we differentiate $s^2 = (x-u)^2 + (y-v)^2$ with respect to t and divide by 2, then we get

$$s\frac{ds}{dt} = (x-u)\left(\frac{dx}{dt} - \frac{du}{dt}\right) + (y-v)\left(\frac{dy}{dt} - \frac{dv}{dt}\right) \ .$$

If we substitute into the above equation the data at 9 pm, then we get

$$20\left(\frac{ds}{dt} \text{ at 9 pm}\right) = (-12)(-32\pi) + (-16)(2\pi).$$

Solving, we obtain the required derivative $\frac{ds}{dt}$ at 9 pm.

(18)(a) Let x be the width of the A picture. The width of the B picture is $1000 - x$. The A picture has size x by $(10/8)x$. The B picture has size $1000 - x$ by $(7/5)(1000 - x)$. We are minimizing the sum of the two areas, and this is $(10/8)x^2 + (7/5)(1000 - x)^2$. We are minimizing over the interval $[0, 1000]$. The mnimum occurs when $x = 28000/53$.

(18)(b) The maximum occurs when $x = 0$. The picture of celebrity A disappears completely, and pet B takes over the entire 1000 pixel width of the blue rectangle.

(18)(c) In parts (a), (b) we did not have to worry about the vertical size of the blue rectangle. Neither picture would extend below the bottom edge of the blue rectangle. In part (c) the situation is different. The new A picture has size x by $2x$. Since the new A picture must stay inside the blue rectangle, we must have $2x \leq 1800$. This says $x \leq 900$. Instead of maximizing over x in the interval $[0, 1000]$, we are actually maximizing over x in the interval $[0, 900]$. The maximum occurs when $x = 900$. Celebrity A does not take over the entire 1000 pixel width of the blue rectangle. There is room left over for a 100 pixel wide picture of pet B in the upper right corner.