138 FINAL EXAM FORMULA SHEET

The future value of the income over the time period T is given by $FV = \int_0^T f(t)e^{r(T-t)}dt$.

The **present value** is given by: $PV = \int_0^T f(t)e^{-rt}dt$

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f"(a)}{2!}(x-a)^2 + ... \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the **nth Taylor Polynomial of the function f at a**.

The remainder Formula: If $|f^{n+1}(x)| \leq M$ to all number between x and a, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Differential Equations

Euler's Method: approximates the values of the solutions for the DE $dy/dx = f(x,y)$ with $y(x_0) = y_0$ at specific points:

$$y_0 = y(x_0), \qquad y_1 = y_0 + hf(x_0, y_0), \qquad ... \qquad y_{n+1} = y_n + hf(x_n, y_n)$$

First order DE: The general solution of a DE of the form $\frac{dy}{dx} + p(x)y = q(x)$ is $\frac{1}{I(x)}\left[\int I(x)q(x)dx + C\right]$ where $I(x) = e^{\int p(x)dx}$

Second Order Homogeneous Linear DE; $\mathbf{ay" + by' + cy = 0 \quad a \neq 0}$

The characteristic equation of $ay" + by' + cy = 0$ is $ar^2 + br + c = 0$.

When the CE has 2 distinct real roots, r_1, r_2, the solution is $y = C_1e^{r_1x} + C_2e^{r_2x}$.

When the CE has 2 equal real roots, $r_1 = r_2 = r$, the solution is $y = (C_1 + C_2x)e^{rx}$.

When the CE has 2 distinct none real (complex) roots, $r_1 = \alpha + \beta i \quad and \quad r_2 = \alpha - \beta i$, the solution is $y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$.

Variation Of Parameters: Let $y_h = C_1y_1 + C_2y_2$ be the solution for the homogeneous DE $ay' + by' + cy = 0$. Then the particular solution for the nonhomogeneous DE $ay'' + by' + cy = F(x)$ is $\mathbf{y_p = uy_1 + vy_2}$ where

$$\mathbf{u(x) = \int \frac{-y_2F(x)}{y_1y_2' - y_2y_1'}dx \qquad and \qquad v(x) = \int \frac{y_1F(x)}{y_1y_2' - y_2y_1'}dx}$$

Note that the solution is $y_h + y_p = C_1y_1 + C_2y_2 + uy_1 + vy_2$

Exponential growth and decay: $\dfrac{dQ}{dt} = kQ(t).$

The Logistics Equation with $\mathbf{Q_0 < L}$:

$$\frac{dQ}{dt} = aQ - kQ^2 \quad or \quad if \quad let \quad L = a/k \quad \frac{dQ}{dt} = kQ(L-Q)\ .$$

The solution of the equation is $Q(t) = \dfrac{L}{1 + Ae^{-at}}$ *and* $A = \dfrac{L}{Q_0} - 1$

Let $Q_0 = Q(0)$, $Q_1 = Q(T)$ *and* $Q_2 = Q(2T)$, then:

$$\frac{\frac{1}{Q_1} - \frac{1}{Q_2}}{\frac{1}{Q_0} - \frac{1}{Q_1}} = e^{-aT}, \qquad \frac{A}{L} = \frac{\frac{1}{Q_0} - \frac{1}{Q_1}}{1 - e^{-aT}} \quad and \quad \frac{1}{L} = \frac{1}{Q_0} - \frac{A}{L}$$

Numerical Integration

Trapezoidal Rule: $\displaystyle\int_a^b f(x)dx \approx T_n = \frac{1}{2}\left(\frac{b-a}{n}\right)[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)]$

Simpson's Rule, n even: $\displaystyle\int_a^b f(x)dx \approx S_n = \frac{1}{3}\left(\frac{b-a}{n}\right)[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+ 2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$

Trapezoidal Rule Error Bound $|E_n| \leq \dfrac{(b-a)^3}{12n^2}M$ when $|f^{"}(x)| \leq M$ for all $a \leq x \leq b$.

Simpson's Rule Error Bound $|E_n| \leq \dfrac{K(b-a)^5}{180n^4}$ when $|f^{(4)}(x)| \leq K$ for all $a \leq x \leq b$

Linear Algebra

Matrix A is **invertible** only if A is a square matrix with nonzero determinant. If A^{-1} exists then $AA^{-1} = A^{-1}A = I$. The (i,j) entry of A^{-1} is $\dfrac{A_{ji}}{detA}$, where $A_{ji} = (-1)^{j+i}M_{ji}$.

Cramer's Rule: Let $A \cdot x = b$. Then $x_i = \dfrac{detB_i}{detA}$, where B_i is the matrix formed from A by replacing in the *ith* column of A with the vector b.

Eigenvalues: If λ is an eigenvalue of A then

$$the\ characteristic\ polynomial\ of\ the\ matrix\ A = det(A - \lambda I) = 0.$$

x is an **eigenvector** of A for the eigenvalue λ if $A \cdot x = \lambda x$.

The eigenvectors of A are also the eigenvectors of A^k and the eigenvalues of A^k are λ^k

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C \qquad \int e^x = e^x + C$$

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int x^n = \frac{x^{n+1}}{n+1} + C$$

Integration by parts: $\displaystyle\int udv = uv - \int vdu \quad and \quad \int_a^b udv = uv|_a^b - \int_a^b vdu$