1A

(10) 1. Suppose $f(x) = 2x^2 - 3x$. Use the **definition of derivative** to find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} = \lim_{h \to 0} 4x + 2h - 3 = 4x - 3$$

(9) 2. Find an equation for the line tangent to the graph of $y = \sqrt{x} + 2x^2$ at the point where $x = 1$.

$y = x^{1/2} + 2x^2$, so $y' = \dfrac{1}{2}x^{-1/2} + 4x$. At $x = 1$, the value y is 3 and the value of y' is $9/2$. Thus an equation for the tangent is

$$y - 3 = \frac{9}{2}(x - 1).$$

(12) 3. Assume that the functions $u(x)$ and $v(x)$ are defined and differentiable for all real numbers x. The following data is known about u, v, and their derivatives.

x	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$
2	3	4	-1	2
3	2	1	3	-1
4	1	3	0	-2

Define $f(x) = u(x)^2 + 2v(x)$ and $g(x) = v(x)/u(x)$. Answer the following, giving a brief explanation of how the answers were obtained.

a) What is $f'(2)$?

Since the chain rule had not been covered when the test was given, to differentiate $u(x)^2$ we have to write it as $u(x)u(x)$ and use the product rule.

$$f'(x) = u(x)u'(x) + u'(x)u(x) + 2v'(x) = 2u(x)u'(x) + 2v'(x).$$

Thus

$$f'(2) = 2(3)(-1) + 2(2) = -2.$$

b) What is $g'(3)$?

$$g'(x) = \frac{u(x)v'(x) - v(x)u'(x)}{u^2(x)}$$

Thus

$$g'(3) = \frac{2(-1) - 1(3)}{2^2} = -\frac{5}{4}.$$

c) What can be said about the number and location of solutions to the equation $f(x) = 6.5$ with x in $[2, 4]$?

From the table, we have $f(2) = 17$, $f(3) = 6$, and $f(4) = 7$. By the Intermediate Value Theorem, there is at least one solution to the equation $f(x) = 6.5$ in the interval $[2, 3]$ and at least one in the interval $[3, 4]$. Thus the total number of solutions is at least 2.

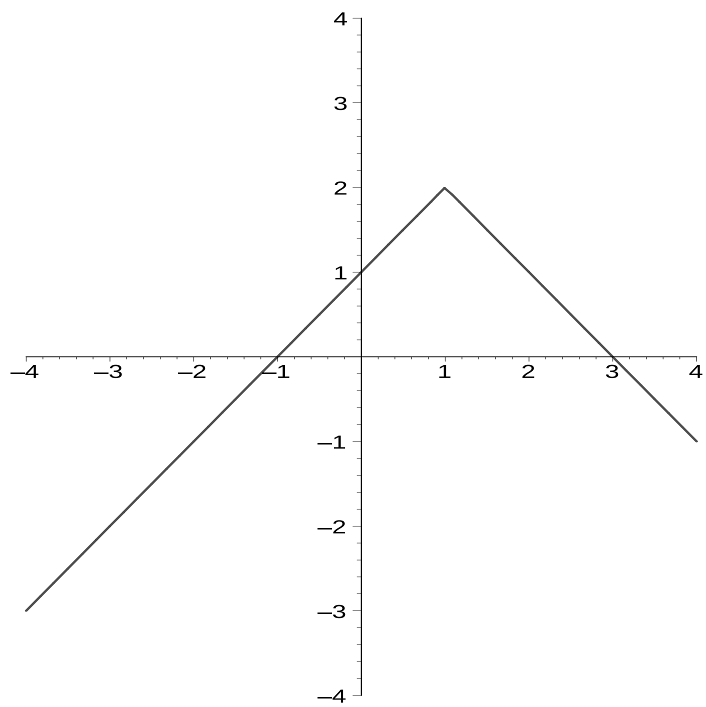
(12) 4. Suppose that the function $f(x)$ is described by

$$f(x) = \begin{cases} x + B & \text{if } x < 1 \\ Ax + 3 & \text{if } x \geq 1 \end{cases}.$$

a) Find A and B so that $f(x)$ is continuous for all numbers and $f(-1) = 0$. Briefly explain your answer.

The only place that f might not be continuous is at $x = 1$, where the definition changes. Now $f(1) = A + 3$ while $\lim_{x \to 1^-} f(x) = 1 + B$. If f is to be continuous at $x = 1$, we must have $A + 3 = 1 + B$.
The value of $f(-1)$ is $-1 + B$, which must be 0. This gives $B = 1$. Substituting this value in the previous equation, we get $A = -1$.

b) Sketch $y = f(x)$ on the axes given for the values of A and B found in a) when x is in the interval $[-2, 2]$.



(16) 5. Evaluate the indicated limits exactly. Give evidence to support your answers without appealing to calculator computations, to graphing, or to l'Hôpital's Rule.

a) $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

This limit is

$$\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

b) $\displaystyle\lim_{x\to 2^-}\frac{|x-1|-1}{|x-2|}$

If x is a real number less than 2 but very close to 2, then $x-2$ will be negative and $x-1$ will be positive. Thus $|x-2|=2-x$ and $|x-1|=x-1$. Thus this limit is

$$\lim_{x\to 2^-}\frac{x-1-1}{2-x}=\lim_{x\to 2^-}-1=-1.$$

c) $\displaystyle\lim_{x\to 0}\frac{\sin^2 2x}{x^2}$

This limit is

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2=\lim_{x\to 0}\left(2\frac{\sin 2x}{2x}\right)^2=4\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)^2=4(1^2)=4.$$

d) $\displaystyle\lim_{x\to 0}\frac{\cos 3x-1}{x}$

This limit is

$$\lim_{x\to 0}3\frac{\cos 3x-1}{3x}=3\lim_{x\to 0}\frac{\cos 3x-1}{3x}=3(0)=0.$$

(14) 6. In the following, distances are measured in feet and time in seconds. A particle is moving on the x-axis. Its position at time t is given by $s(t)=2t^3-3t^2-12t+7$.

a) What is the net distance traveled by the particle from $t=1$ to $t=3$?

The net distance for this trip is $|s(3)-s(1)|=|-2-(-6)|=4$.

b) What is the total distance traveled by the particle from $t=1$ to $t=3$?

If the particle does not reverse direction, then the total distance is the same as the net distance. Thus we need to see if the particle reverses direction. At a reversal, the velocity is momentarily 0. The velocity is

$$v(t)=s'(t)=6t^2-6t-12=6(t^2-t-2)=6(t+1)(t-2).$$

Setting $v(t) = 0$, we get $t = -1$ or $t = 2$. The value $t = -1$ is not relevant for our question, since it is not between 1 and 3. However, $t = 2$ is relevant. We find that $s(2) = -13$. The total distance traveled is

$$|s(2) - s(1)| + |s(3) - s(2)| = | - 13 - (-6)| + | - 2 - (-13)| = 7 + 11 = 18.$$

(10) 7. Solve the following two equations for x.

a) $4^{2x-3} = 8^{x+1}$

Taking the logarithm to the base 2 of both sides, we get

$$(2x - 3)\log_2 4 = (x + 1)\log_2 8.$$

Since $\log_2 4 = 2$ and $\log_2 8 = 3$, this gives

$$2(2x - 3) = 3(x + 1).$$

The single solution of this equation is $x = 9$.

b) $\ln(x - 2) + \ln(x + 1) = \ln(3x - 2)$

The left side is $\ln[(x - 2)(x + 1)]$, so raising e to both sides gives

$$(x - 2)(x + 1) = 3x - 2,$$

or $x^2 - 4x = 0$. The solutions to this quadratic equation are $x = 0$ and $x = 4$. The value $x = 0$ is not legal for the original equation, since $\ln -2$ is not defined. The value $x = 4$ is the only solution to the original equation.

(8) 8. (There is no single correct answer to this problem.) On the axes below, sketch the graph of a function $f(x)$ with all the following properties:

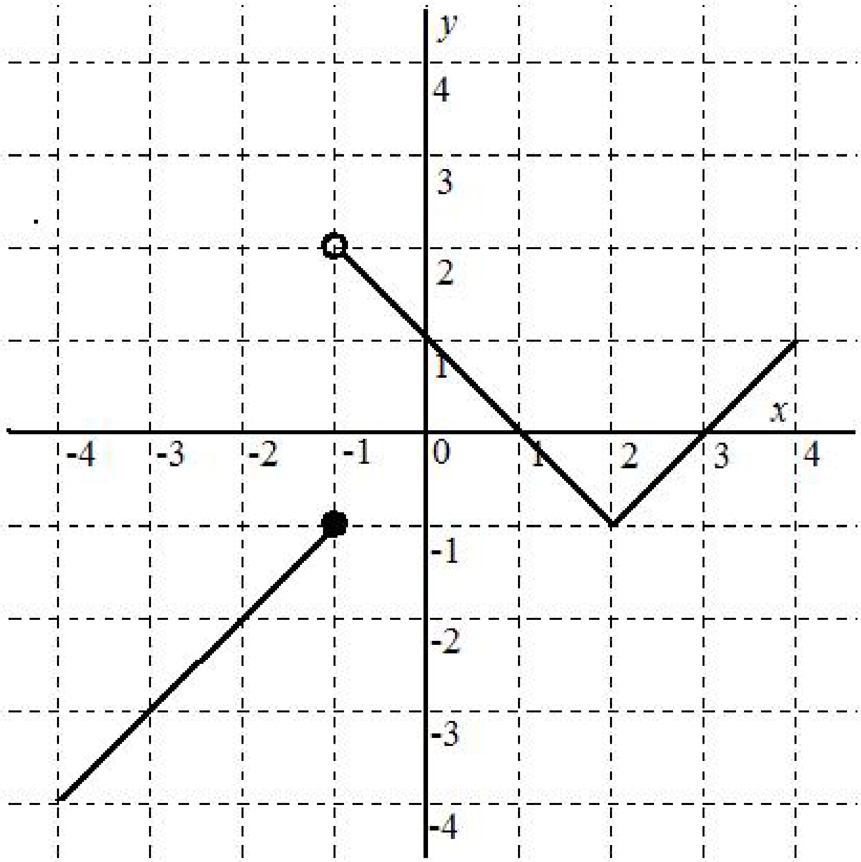
a) The domain of $f(x)$ is $[-4, 4]$.

b) $f(x)$ is differentiable at all points of its domain except $x = -1$ and $x = 2$.

c) $f(x)$ is not continuous at $x = -1$.

d) $f(x)$ is continuous but not differentiable at $x = 2$.

e) $f(0) = 1$ and $f'(0) = -1$.



(9) 9. a) If $f(x) = 2x^2\sqrt{x} + \dfrac{3}{x^3\sqrt{x}}$, what is $f'(x)$?

$$f(x) = 2x^{5/2} + 3x^{-7/2}.$$

Thus

$$f'(x) = 2\left(\frac{5}{2}\right)x^{3/2} + 3\left(\frac{-7}{2}\right)x^{-9/2}.$$

b) If $f(x) = \dfrac{2\tan x - 3\sec x}{\ln x}$, what is $f'(x)$?

$$f'(x) = \frac{(\ln x)(2\sec^2 x - 3\sec x\tan x) - (2\tan x - 3\sec x)\left(\frac{1}{x}\right)}{\ln^2 x}.$$

c) If $f(x) = xe^x\sin x$, what is $f'(x)$?

Since $f(x)$ is a product of three factors, we have to use the product rule twice. Here is one way to do the problem:

$$f'(x) = (xe^x)\cos x + (xe^x)'\sin x = (xe^x)\cos x + (xe^x + e^x)\sin x.$$