(16) 1. Calculate the following limits. Give a brief justification of your answers without reference to calculator computations or graphing.

(a) $\lim\limits_{x\to 0} \dfrac{4x}{\sin 3x}$

$$\lim_{x\to 0} \frac{4x}{\sin 3x} = \lim_{x\to 0} \frac{\frac{4}{3}}{\frac{\sin 3x}{3x}} = \frac{\frac{4}{3}}{1} = \frac{4}{3}$$

(b) $\lim\limits_{x\to -\infty} \dfrac{3x^4 + x^3 - 2x^2 + 10}{7x^5 - 4x^3 + 2x + 1}$

$$\lim_{x\to -\infty} \frac{3x^4 + x^3 - 2x^2 + 10}{7x^5 - 4x^3 + 2x + 1} = \lim_{x\to -\infty} \frac{\frac{3}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{10}{x^5}}{7 - \frac{4}{x^2} + \frac{2}{x^4} + \frac{1}{x^5}} = \frac{0}{7} = 0$$

(c) $\lim\limits_{x\to 0} \dfrac{\cos(2x) - 1}{x^2}$

Using l'Hôpital's Rule twice,

$$\lim_{x\to 0} \frac{\cos(2x) - 1}{x^2} = \lim_{x\to 0} \frac{-2\sin(2x)}{2x} = \lim_{x\to 0} \frac{-4\cos(2x)}{2} = \frac{-4}{2} = -2$$

(d) $\lim\limits_{x\to -3^-} \dfrac{|2x+6|}{x+3}$

If x is slightly less than -3, then $2x + 3 < 0$, so $|2x + 6| = -2x - 6$. Thus

$$\lim_{x\to -3^-} \frac{|2x+6|}{x+3} = \lim_{x\to -3^-} \frac{-2(x+3)}{x+3} = \lim_{x\to -3^-} -2 = -2$$

(10) 2. Compute the derivative of $\dfrac{2}{x-1}$ **directly from the definition.**

$$\left(\frac{2}{x-1}\right)' = \lim_{h\to 0} \frac{\frac{2}{x+h-1} - \frac{2}{x-1}}{h} = \lim_{h\to 0} \frac{2(x-1) - 2(x+h-1)}{h(x+h-1)(x-1)} =$$

$$\lim_{h\to 0} \frac{-2h}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = \frac{-2}{(x-1)^2}$$

(16) 3. Compute the derivatives with respect to x of the following functions. Algebraic simplification of the answers need not be performed.

(a) $e^x \ln(2x)$

$$e^x\left(\frac{1}{2x}\right)2 + e^x \ln(2x)$$

(b) $\dfrac{\sin x}{x^3 + 2x}$

$$\frac{(x^3+2x)\cos(x) - \sin(x)(3x^2+2)}{(x^3+2x)^2}$$

(c) $\displaystyle\int_x^0 \sec t\, dt$

$$\left(\int_x^0 \sec t\, dt\right)' = \left(-\int_0^x \sec t\, dt\right)' = -\sec x$$

(d) $\displaystyle\int_0^{x^2+x} \sin(2t)\, dt$

$$\sin(2(x^2+x))(2x+1)$$

(10) 4. Suppose that f is a function with first and second derivatives. Suppose in addition that the following values are known: $f(1) = 0$, $f'(1) = 3$, and $f''(1) = 5$. If $g(x) = e^{f(x)}$, what are $g'(1)$ and $g''(1)$?

$$g'(x) = e^{f(x)} f'(x)$$

so $g'(1) = e^0(3) = 3$.

$$g''(x) = e^{f(x)} f''(x) + e^{f(x)} f'(x)^2$$

Thus $g''(1) = e^0(5) + e^0(3)^2 = 14$

(15) 5. Find the following indefinite integrals:

(a) $\displaystyle\int \left(x^4 + \sec(x)\tan(x) - \frac{2}{x}\right)\, dx$

$$\frac{x^5}{5} + \sec x - 2\ln|x| + C$$

(b) $\displaystyle\int 3x^2 \sin(x^3+1)\, dx$

If $u = x^3 + 1$, then $du = 3x^2 dx$ and

$$\int 3x^2 \sin(x^3+1)\, dx = \int \sin(u) du = -\cos(u) + C = -\cos(x^3+1) + C$$

(c) $\displaystyle\int \cos(x) e^{\sin(x)}\, dx$

If $u = \sin(x)$, then $du = \cos(x) dx$ and

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^u\, du = e^u + C = e^{\sin(x)} + C$$

(18) 6. Compute the following:

(a) $\displaystyle\int_1^2 \frac{x^3+5}{x}\, dx$

$$\int_1^2 \frac{x^3+5}{x}\, dx = \int_1^2 \left(x^2 + \frac{5}{x}\right)\, dx = \left(\frac{x^3}{3} + 5\ln x\right)\Big|_1^2 = \frac{8}{3} + 5\ln 2 - \left(\frac{1}{3} + 5\ln 1\right) =$$

$$\frac{7}{3} + 5\ln 2$$

(b) The area under the graph of $y = 4x + e^x$ on the interval $[0, 2]$.

The area is

$$\int_0^2 (4x + e^x)\, dx = (2x^2 + e^x)|_0^2 = 8 + e^2 - 1 = 7 + e^2$$

(c) $\displaystyle\int_0^1 \frac{x}{(x^2+1)^2}\,dx$

If $u = x^2 + 1$, then $du = 2x dx$, so

$$\int_0^1 \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_1^2 \frac{du}{u^2} = \frac{1}{2}\int_1^2 u^{-2}\,du = \frac{1}{2}(-u^{-1})|_1^2 = 1/2\left(-\frac{1}{2}+1\right) = \frac{1}{4}$$

(18) 7. In the following, A and B are constants. Let f be the function defined by

$$f(x) = \begin{cases} x^2 + Ax + 3 & \text{if } x < -1 \\ x^3 & \text{if } -1 \le x < 1 \\ Bx^2 + 2x & \text{if } 1 \le x \end{cases}.$$

(a) What are $f(-1)$ and $f(1)$?

$f(-1) = -1$ and $f(1) = B + 2$.

(b) What are $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to 1^-} f(x)$?

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 + Ax + 3 = -A + 4$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 = 1$$

(c) Assume that $f(x)$ is continuous for all x. What are A and B?

If $f(x)$ is continuous at -1 and 1, then $-1 = -A + 4$ and $B + 2 = 1$. From this it follows that $A = 5$ and $B = -1$.

(d) Suppose that A and B have the values determined in (c). Is $f(x)$ differentiable for all x?

The slope of the tangent to x^3 is $3x^2$, which at $x = 1$ is 3. The slope of the tangent to $-x^2 + 2x$ is $-2x + 2$, which at $x = 1$ is 0. Since the slopes are not the same, the graph has a sharp corner at $x = 1$ and so the function is not differentiable there.

(8) 8. Let $f(x) = 3x + \ln(x^2 - 3)$. Use the linearization of $f(x)$ at $x = 2$ to estimate the value of $f(1.97)$.

$f(2) = 6$ and $f'(x) = 3 + \dfrac{2x}{x^2-3}$, so $f'(2) = 7$. The linearization at $x = 2$ is $L(x) = 6 + 7(x - 2)$. The value of $L(1.97)$ is $6 + 7(-.03) = 5.79$.

(12) 9. Let $f(x) = 3x^5 - 5x^4 - 20x^3 + 13x - 3$. At which point of inflection of the graph of $f(x)$ does the tangent line have positive slope?

$f'(x) = 15x^4 - 20x^3 - 60x^2 + 13$ and $f''(x) = 60x^3 - 60x^2 - 120x = 60x(x^2 - x - 2) = 60x(x-2)(x+1)$. Thus $f''(x) = 0$ for $x = -1$, 0, and 2. The value of $f''(x)$ changes sign at each of these values, so there are three points of inflection. Now $f'(-1) = -12$, $f'(0) = 13$, and $f'(2) = -147$. Thus only at $x = 0$ is the slope of the tangent positive.

(16) 10. A particle is moving along the x-axis in such a way that its position, measured in feet, is

$$s(t) = 6t^4 + 12t^3 - 72t^2 + 12t - 7,$$

where t is the time in seconds. During the interval $-1 \leq t \leq 2$, what are the absolute maximum and the absolute minimum values of the velocity of the particle?

Let $v(t)$ be the velocity of the particle and let $a(t) = v'(t)$ be the acceleration. Then $v(t) = 24t^3 + 36t^2 - 144t + 12$. The critical numbers of $v(t)$ are the values of t for which $a(t)$ is 0 or undefined. Now $a(t) = 72t^2 + 72t - 144 = 72(t^2 + t - 2) = 72(t+2)(t-1)$. The critical numbers of $v(t)$ are 1 and -2, but only 1 is in the interval $[-1, 2]$. Since $v(-1) = 168$, $v(1) = -72$, and $v(2) = 60$, the absolute maximum value of v on the interval is 168 ft/sec and the absolute minimum of v is -72 ft/sec.

(8) 11. Compute the value of the Riemann sum for the function x^2 on the interval $[-2, 4]$ using the partition -2, 0, 2, 4 and taking as the representative points the midpoint of each subinterval.

The small intervals all have length 2. The representative points are $x_1^* = -1$, $x_2^* = 1$, and $x_3^* = 3$. The Riemann sum is

$$R = [(-1)^2 + 1^2 + 3^2]2 = 22$$

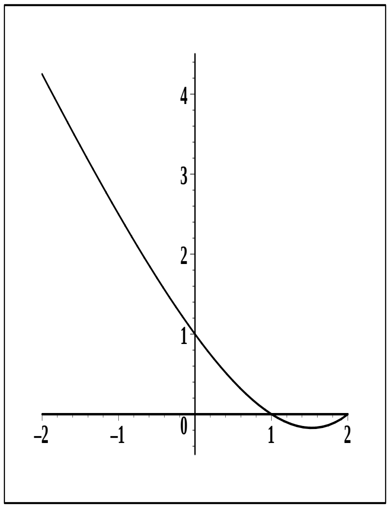
(15) 12. An apartment complex has 90 units. When the monthly rent for each unit is \$600, all units are occupied. Experience indicates that for each \$20-per-month increase in rent, 3 units will become vacant. Each rented apartment costs the owners of the complex \$320 per month to maintain. What monthly rent should be charged to maximize the owners' profits?

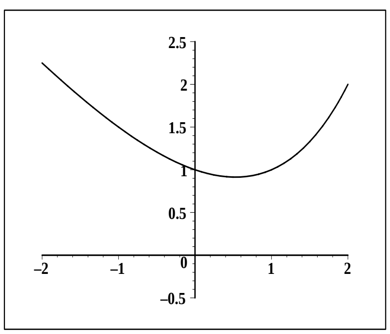
Let x be the number of \$20-per-month increases in the rent. The number of rented units will be $90 - 3x$ and the rent paid for each of these units will be $600 + 20x$. The profit on one unit will be $600 + 20x - 320 = 280 + 20x$. The total profit will be

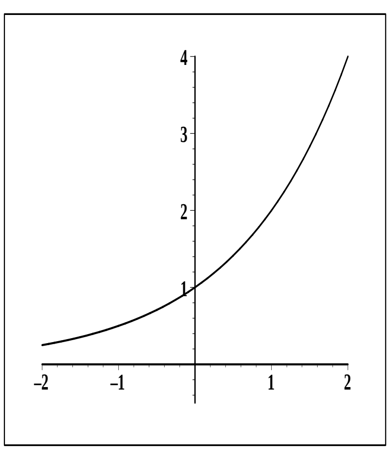
$$P = (90 - 3x)(280 + 20x) = 25200 + 960x - 60x^2$$

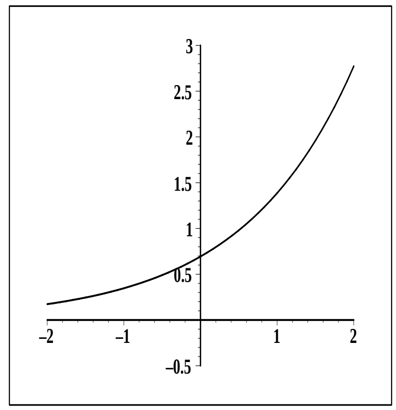
$P' = 960 - 120x = 120(8 - x)$. Thus $x = 8$ is the only critical number of P. The only meaningful values of x are $x \geq 0$. When $x = 0$, the value of P is \$25200 and when $x = 8$ the value of P is \$29,040. After $x = 8$, the value of P decreases, so the profit is maximized when the rent is \$760.

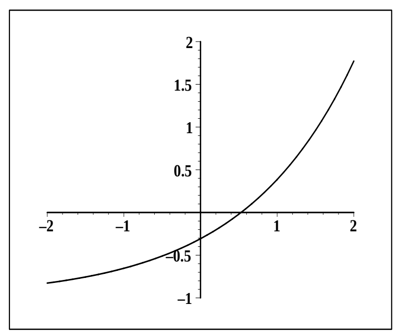
(10) 13. On the left below are the graphs of three functions and on the right are the graphs of their derivatives in some order. Draw a line from each graph on the left to its derivative on the right.

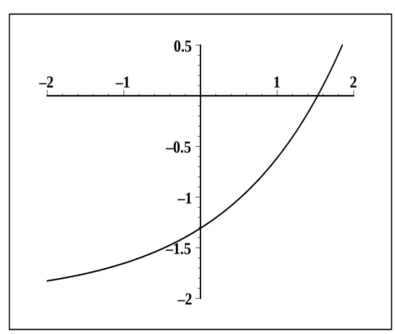












The first graph on the left is connected to the third graph on the right.
Th second graph on the left is connected to the second graph on the right.
The third graph on the left is connected to the first graph on the right.

(18) 14. In the following, assume that distances are measured in feet and time in seconds. A particle is moving along the parabola with equation $y = x^2$. At the moment the particle reaches the point $(1, 1)$, the x-coordinate of the point is increasing at the rate of 3 feet/second.

a) At this moment, what is the rate of change of the y-coordinate of the point?

Differentiating both sides of the equation $y = x^2$ with respect to t, we get $y' = 2xx'$. At moment in question, $y' = 2(1)(3) = 6$ ft/sec.

b) At this moment, how far is the particle from the point $(5, -2)$?

Let z denote the distance of the point from $(5, -2)$. Then $z^2 = (x - 5)^2 + (y + 2)^2$. At the moment, $z^2 = 4^2 + 3^2 = 25$, so $z = 5$.

c) At this moment, how fast is the distance of the particle from the point $(5, -2)$ changing? Is the distance increasing or decreasing?

Differentiating both sides of the equation determining z, we get $2zz' = 2(x-5)x'+2(y+2)y'$. At the moment, $10z' = -8x' + 6y' = -8(3) + 6(6) = 12$, so $z' = 1.2$ ft/sec. The distance z is increasing.

(10) 15. On the axes below, sketch the graph of a function f with the following properties: $f(x)$ is defined and differentiable for all real numbers x except $x = -1$ and $x = 3$. The graph of f has vertical asymptotes at $x = -1$ and $x = 3$.

$$\lim_{x\to\infty} f(x) = -1 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 2.$$

The function f is increasing on the interval $(-1, 3)$. There is a point of inflection at $x = 1$. The graph is concave up on $(-\infty, -1)$. There is a relative maximum at $x = 5$.