(25) 1. Calculate the following limits. Give a brief justification of your answers without reference to calculator computations or graphing.

(a) $\lim_{x\to 3} \frac{x^2-9}{x-3}$

First solution:

$$\lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3} x+3 = 6$$

Second solution using L'Hôpital's Rule:

$$\lim_{x\to 3} \frac{2x}{1} = 6$$

(b) $\lim_{x\to 0} \frac{\sin 3x}{\tan 2x}$

First solution:

$$\lim_{x\to 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}} = \lim_{x\to 0} \cos 2x \left(\frac{3}{2}\right) \frac{\frac{\sin 3x}{3x}}{\frac{\sin 2x}{2x}} = 1\left(\frac{3}{2}\right)\left(\frac{1}{1}\right) = \frac{3}{2}$$

Second solution:

$$\lim_{x\to 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 3x}} = \lim_{x\to 0} \cos 2x \frac{\sin 3x}{\sin 2x} = \frac{3}{2}$$

Third solution using L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{3\cos 3x}{2\sec^2 2x} = \frac{3}{2}$$

(c) $\lim_{x\to\infty} \frac{5x^4+7x^3+2x^2+10}{7x^4+5x^3+2x+1}$

$$\lim_{x\to\infty} \frac{5+\frac{7}{x}+\frac{2}{x^2}+\frac{10}{x^4}}{7+\frac{5}{x}+\frac{2}{x^3}+\frac{1}{x^4}} = \frac{5}{7}$$

(d) $\lim_{x\to\infty} \frac{e^{x/2}}{x^2}$

Applying L'Hôpital's Rule once, we get

$$\lim_{x\to\infty} \frac{\frac{1}{2}e^{x/2}}{2x},$$

which is still indeterminate. Applying the rule again, we get

$$\lim_{x\to\infty} \frac{\frac{1}{4}e^{x/2}}{2} = \infty.$$

Thus the original limit is $\infty$.

(e) $\lim_{x\to 3} \frac{x^2}{\ln x}$

$$\frac{3^2}{\ln 3}$$

(10) 2. Compute the derivative of $\sqrt{x+3}$ **directly from the definition**.

$$\left(\sqrt{x+3}\right)' = \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h+3} - \sqrt{x+3}\right)\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{x+h+3-(x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}$$

(25) 3. Compute the derivatives with respect to $x$ of the following functions. Algebraic simplification of the answers need not be performed.

(a) $\ln(x)\cos(2x)$

$$\frac{\cos(2x)}{x} - 2\ln(x)\sin(2x)$$

(b) $\dfrac{e^x}{2x^3+x}$

$$\frac{(2x^3+x)e^x - e^x(6x^2+1)}{(2x^3+x)^2} = \frac{e^x(2x^3-6x^2+x-1)}{(2x^3+x)^2}$$

(c) $\displaystyle\int_0^x \sec t\,dt$

By the Fundamental Theorem of Calculus, the derivative is

$$\sec x$$

(d) $\displaystyle\int_0^{x^3} e^{t^2}\,dt$

By the the Fundamental Theorem of Calculus plus the chain rule, the derivative is

$$e^{(x^3)^2}3x^2 = e^{x^6}3x^2$$

(e) $\sqrt{x^4+3}$

$$\frac{1}{2}(x^4+3)^{-1/2}4x^3 = \frac{2x^3}{\sqrt{x^4+3}}$$

(10) 4. Suppose that $f$ is a function with first and second derivatives. Suppose in addition that the following values are known: $f'(0)=2$, $f'(1)=3$, $f''(0)=4$, and $f''(1)=5$. If $g(x)=f(\ln(x))$, what are $g'(1)$ and $g''(1)$?

$$g'(x) = f'(\ln x)/x \quad \text{so} \quad g'(1) = f'(\ln 1)/1 = f'(0) = 2$$

$$g''(x) = \frac{xf''(\ln x)/x - f'(\ln x)}{x^2} = \frac{f''(\ln x) - f'(\ln x)}{x^2} \quad \text{so} \quad g''(1) = \frac{4-2}{1} = 2$$

(15) 5. Find the following indefinite integrals:

(a) $\int (x^3 + \frac{3}{x} + \cos x)\, dx$

$$\frac{x^4}{4} + 3\ln|x| + \sin x + C$$

(b) $\int (2x+1)\sec^2(x^2+x)\, dx$

Making the substitution $u = x^2 + x$ and $du = (2x+1)\,dx$, we get

$$\int \sec^2 u\, du = \tan u + C = \tan(x^2 + x) + C$$

(c) $\int \frac{6x^2 - 4}{(x^3 - 2x + 1)^3}\, dx$

Making the substitution $u = x^3 - 2x + 1$ and $du = (3x^2 - 2)\,dx$, we get

$$2\int u^{-3}\, du = \frac{2u^{-2}}{-2} + C = -\frac{1}{(x^3 - 2x + 1)^2} + C$$

(18) 6. Compute the following:

(a) $\int_1^2 \frac{\sqrt{x} + 8}{x}\, dx$

$$\int_1^2 (x^{-1/2} + 8x^{-1})\, dx = \frac{x^{1/2}}{1/2} + 8\ln(x)\,\Big|_1^2 = 2\sqrt{2} + 8\ln 2 - 2$$

(b) The area under the graph of $y = 2 + x^2 + \sin x$ on the interval $[0, \pi]$.

$$\int_0^\pi (2 + x^2 + \sin x)\, dx = 2x + \frac{x^3}{3} - \cos x\,\Big|_0^\pi = 2\pi + \frac{\pi^3}{3} - (-1) - (-1) = 2 + 2\pi + \frac{\pi^3}{3}$$

(c) $\int_0^\pi x^2 \sin(x^3)\, dx$

If we make the substitution $u = x^3$ in this integral, we have $du = 3x^2\, dx$ or $x^2\, dx = du/3$. The integral becomes

$$\frac{1}{3}\int_0^{\pi^3} \sin(u)\, du = -\frac{1}{3}\cos(u)\,\Big|_0^{\pi^3} = -\frac{1}{3}\cos(\pi^3) - \left(-\frac{1}{3}\right) = \frac{1}{3}(1 - \cos(\pi^3))$$

(10) 7. In the following, $A$ and $B$ are constants. Let $f$ be the function defined by

$$f(x) = \begin{cases} x^3 + Ax & \text{if } x \le 1 \\ Bx^2 + 2 & \text{if } x > 1 \end{cases}.$$

(a) What is $\lim_{x \to 1^-} f(x)$?

$$A + 1$$

(b) What is $\lim_{x \to 1^+} f(x)$?

$$B + 2$$

(c) How must $A$ and $B$ be related if $f(x)$ is continuous at $x = 1$?

$$A + 1 = B + 2 \quad \text{or} \quad A = B + 1.$$

(d) What must the values of $A$ and $B$ be if $f(x)$ is differentiable at $x = 1$?

The derivative of $x^3 + Ax$ is $3x^2 + A$, which has the value $A + 3$ if $x = 1$. The derivative of $Bx^2 + 2$ is $2Bx$, which has the value $2B$ at $x = 1$. If $f(x)$ is differentiable at $x = 1$, these two values must agree. That is, we must have $A + 3 = 2B$. By (c), $A = B + 1$, so $B + 4 = 2B$ or $B = 4$. From this, we get $A = 5$.

(9) 8. Use the linearization of $\tan x$ at $x = \pi/4$ to estimate the value of $\tan(\pi/4 + 0.13)$.

The value of $\tan x$ at $x = \pi/4$ is 1. The derivative of $\tan x$ is $\sec^2 x$, which has value 2 at $x = \pi/4$. Thus the linearization of $\tan x$ at $\pi/4$ is

$$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right) \quad \text{and} \quad L\left(\frac{\pi}{4} + 0.13\right) = 1 + 2(0.13) = 1.26.$$

(10) 9. Find an equation for the tangent line to the graph of $2x^3y^2 + x^2y^3 = 16$ at the point $(1, 2)$.

Differentiating both sides of the given equation with respect to $x$, we get

$$2x^3(2y)y' + 6x^2y^2 + x^2(3y^2)y' + 2xy^3 = 0 \quad \text{or} \quad y' = -\frac{6x^2y^2 + 2xy^3}{4x^3y + 3x^2y^2}.$$

At the point $(1, 2)$, we have

$$y' = -\frac{24 + 16}{8 + 12} = -2,$$

and the tangent is $y - 2 = -2(x - 1)$.

(10) 10. In this problem, assume that coordinates are given in feet. A point is moving along the $x$-axis in such a way that its acceleration at time $t$ is $t + \cos 2t$ ft/sec$^2$.

(a) Suppose the velocity of the point at $t = 0$ is 3 ft/sec. Describe the velocity of the point as a function of $t$.

Let $a(t)$ denote the acceleration. Then the velocity $v(t)$ is an antiderivative of $a(t)$. Thus

$$v(t) = \int (t + \cos(2t))\, dt = \frac{t^2}{2} + \frac{\sin(2t)}{2} + C.$$

But $3 = v(0) = C$, so

$$v(t) = \frac{t^2}{2} + \frac{\sin(2t)}{2} + 3.$$

(b) Suppose the coordinate of the point at $t = 0$ is 10. Describe the position of the point at time $t$.

The position $s(t)$ is an antiderivative of $v(t)$. Hence

$$s(t) = \int \left(\frac{t^2}{2} + \frac{\sin(2t)}{2} + 3\right) dt = \frac{t^3}{6} - \frac{\cos(2t)}{4} + 3t + C.$$

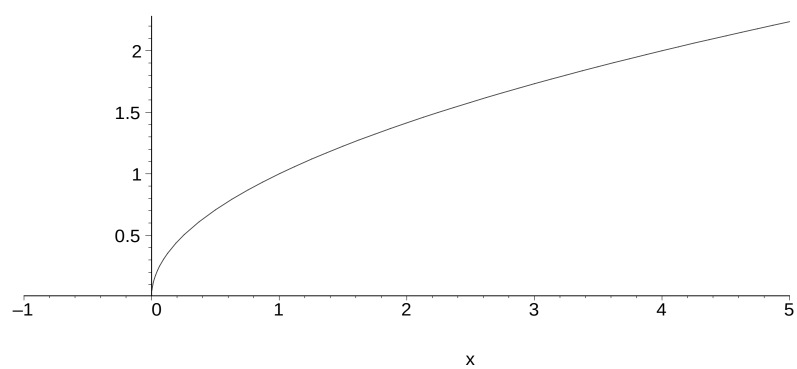
Since $10 = s(0) = -1/4 + C$, we have $C = 41/4$ and

$$s(t) = \frac{t^3}{6} - \frac{\cos(2t)}{4} + 3t + \frac{41}{4}.$$

(8) 11. Compute the value of the Riemann sum for the function $2^x$ on the interval $[-1, 2]$ using the partition -1, 0, 1, 2 and taking as the representative points the right endpoint of each subinterval.

The Riemann sum is $2^0 \cdot 1 + 2^1 \cdot 1 + 2^2 \cdot 1 = 7$.

(10) 12. What point on the graph of $y = \sqrt{x}$ is closest to the point $(3, 0)$?



A typical point on the graph is $P = (x, \sqrt{x})$, where $x \geq 0$. Let $f(x)$ denote the square of the distance from $P$ to $(3, 0)$. (To minimize the distance, it is enough to minimize the square of the distance.) Then

$$f(x) = (x-3)^2 + (\sqrt{x})^2 = (x-3)^2 + x = x^2 - 5x + 9.$$

Thus $f'(x) = 2x - 5$, which is 0 when $x = 5/2$. The first derivative test tells us that $f(x)$ as a relative minimum at $x = 5/2$. There is one endpoint, namely $x = 0$, but $f(0) = 9 > 2.75 = f(5/2)$. Thus $(5/2, \sqrt{5/2})$ is the closest point on the graph to $(3, 0)$.

(10) 13. Find equations for all horizontal and vertical asymptotes of the function $\dfrac{4e^{-x}+3}{7e^{-x}-2}$.

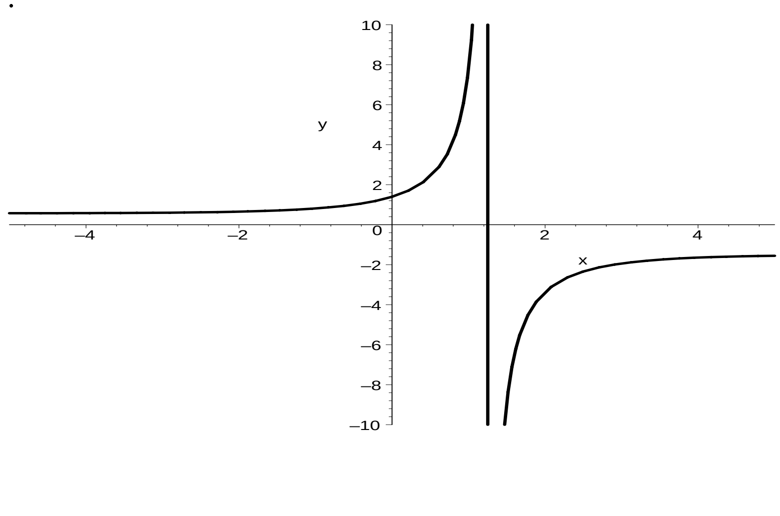
If $f(x)$ is the given function, then

$$\lim_{x\to\infty} f(x) = \frac{4\cdot 0 + 3}{7 \cdot 0 - 2} = -\frac{3}{2} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4 + 3e^x}{7 - 2e^x} = \frac{4 + 3\cdot 0}{7 - 2\cdot 0} = \frac{4}{7}.$$

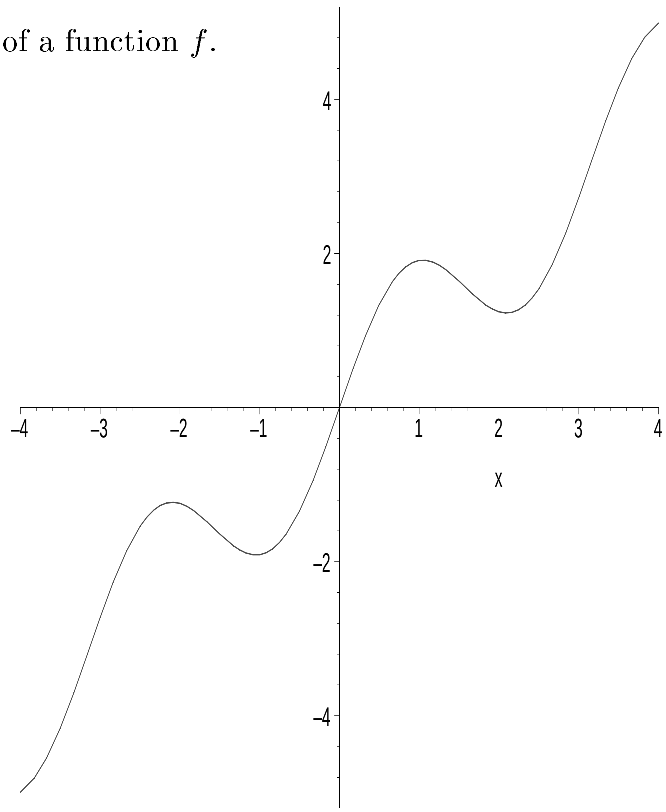
Thus there are two horizontal asymptotes, $y = -3/2$ and $y = 4/7$. The numerator of $f(x)$ is always defined, so a vertical asymptote can only occur when the deminator is 0. If $7e^{-x} - 2 = 0$, then

$$7e^{-x} = 2, \quad e^{-x} = 2/7, \quad -x = \ln(2/7), \text{ or } x = -\ln(2/7) = \ln(7/2).$$

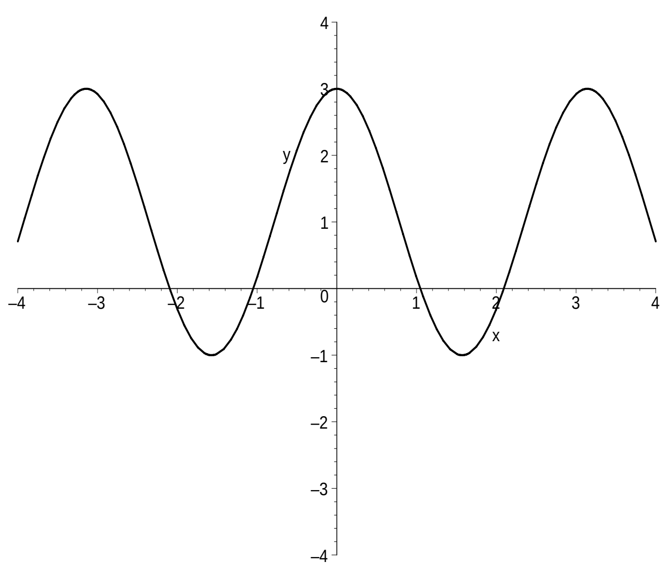
The numerator is not 0 for this value of $x$, so the single vertical asymptote is $x = \ln(7/2)$. Here is a graph of $f$.



(10) 14. Here is the graph of a function $f$.

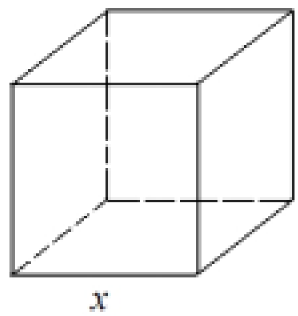



On the axes below, sketch the graph of the derivative of $f$.




(The original graph is actually $x + \sin 2x$ and the derivative is $1 + 2\cos 2x$.)

(10) 15. You may find it hard, but imagine you are watching a balloon in the shape of a cube being inflated. At a certain moment the volume of the balloon is 8 cubic feet and the volume is increasing at the rate of 0.3 cubic feet per minute. How fast is the surface area of the balloon increasing at that moment?




Let $x$ be the length of an edge of the cube. Then the volume of the cube is $V = x^3$. The cube has six faces, each of which is a square with edge $x$. Thus the surface area of the cube is $A = 6x^2$. At the moment, $V(x) = 8$, that is, $x^3 = 8$ or $x = 2$ feet. In general,

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \quad \text{and} \quad \frac{dA}{dt} = 12x\frac{dx}{dt}.$$

At the moment,

$$0.3 = 3 \cdot 4\frac{dx}{dt} \quad \text{so} \quad \frac{dx}{dt} = 0.025 \text{ ft/min.}$$

Thus

$$\frac{dA}{dt} = 12 \cdot 2(0.025) = 0.6 \text{ ft}^2\text{/min.}$$

(10) 16. In the space below, sketch the graph of a function $f$ with the following properties: $f(x)$ is defined and differentiable for all real numbers $x$ except $x = -3$ and $x = 2$. The graph of $f$ has vertical asymptotes at $x = -3$ and $x = 2$.

$$\lim_{x\to\infty} f(x) = -1 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 2.$$

The graph of $f$ is concave down on the intervals $(-\infty, -3)$ and $(2, \infty)$ and the graph is concave up on the interval $(-3, 2)$.