Antiderivatives. We can no pretend that you know how to differentiate any function given by a formula that you know how to write. You may not always get the right answer, but you have seen all the rules that are needed. There are no new ideas in differentiation, although more practice may be needed to master the rules we have. To add to our understanding of the principles of calculus, it is time to ask about recovering a function from its derivative. That is, given a function $f(x)$, find functions $F(x)$ with $F'(x) = f(x)$.

There are many solutions. Suppose $F'(x) = f(x)$ and C is a constant. Then $G(x) = F(x) + C$ has derivative

$$\begin{aligned} \frac{d}{dx}G(x) &= \frac{d}{dx}\big(F(x)\big) + \frac{d}{dx}C \\ &= \frac{d}{dx}\big(F(x)\big) = f(x). \end{aligned}$$

There are no more solutions. If $F'(x) = G'(x) = f(x)$, then $F(x) - G(x)$ has derivative zero. It follows from the mean value theorem that any function with

derivative zero is constant (if the function isn't constant, some difference quotient isn't zero, but all difference quotients are equal to the value of the derivative somewhere).

A new name and notation. The antiderivative is usually called the **indefinite integral** and denoted

$$\int f(x)\,dx.$$

The significance of this notation will be explored in later sections. In particular, although the dx appears to be only an ornament if the only variable in the problem is x, it will turn out to remarkably useful.

Linearity. Suppose that we know integrals of $f(x)$ and $g(x)$. That is, we have functions $F(x)$ and $G(x)$ with $F'(x) = f(x)$ and $G'(x) = g(x)$. Then,

$$\frac{d}{dx}\big(F(x) + G(x)\big) = f(x) + g(x).$$

In the notation of integrals, this can be written

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

In the expression on the left, it is clear that the intent is to integrate the sum since the that is the expression between the integral sign and the symbol dx. Thus one role of dx is to mark the end of the expression to be integrated. This avoids introducing parentheses to hold the expression together.

More generally, if a and b are **constants**,

$$\int af(x)+bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

There is one minor annoyance here. Each of the integrals on the right is only defined up to a term of the form $+C$, and the two terms have **different** and **independent** choices of C, but on the left side, these have been combined into yet another C. It is common to include a $+C$ every time you evaluate an integral, but its only significance is to give the **general solution** after you have found one particular antiderivative. You will be expected to write it to signify that you have finished, but the main part of the problem will be finding **one** solution of $F'(x) = f(x)$.

Polynomials. Since the derivative of x^n is a constant multiple of x^{n-1}, the integral of a power of x will be a **constant multiple** of **one higher power** of x (with one important exception). I used a vague statement for a reason. Although a fair number of integrals will be met in this course, they appear as an extension of the properties of the derivative, and derivatives have taken most of our time. It is better not to introduce a new list of formulas that resemble differentiation formulas just enough to make the differentiation formulas harder to remember. A simpler method for finding integrals involves making a table of pairs $F(x)$, $F'(x)$ for all functions $F(x)$ that you find relevant. If you can discover all functions that appear in the integral of $f(x)$, linearity leads to a system of equations for the coefficients in the expression. In many cases this system of equations will be so easy to solve that it can be solved without even writing it. For example, if you can correctly identify the term whose derivative is the highest degree term of a polynomial, you have reduced the problem to integrating a polynomial of lower degree.