Introduction to exponential models. The exponential function arises naturally in many applications as a consequence of the following property of the derivative of $y = Ae^{ct}$.

$$\frac{dy}{dt} = \frac{d}{dt} Ae^{ct} = A(e^{ct})c = cy.$$

This says that the change in the amount y is proportional to y. This is the expected behavior of population growth (with $c > 0$) or radioactive decay (with $c < 0$).

It can also be shown that these exponential functions are the only functions with this property.

Population. It is a reasonable assumption in building a model of population growth to expect that a certain fraction of the population will reproduce over a standard unit of time. This leads to an equation of the form $dy/dt = cy$. Adjustments in this law, such as including a death rate, do not change the general form of the equation. However, it is a consequence of this equation that $y = AE^{ct}$. Hence, as long as $c > 0$ (i.e., as long as the birth rate is larger than the death

rate) y will eventually become very large while increasing rapidly. A few measurements can determine all parameters in this model and give good predictions for as long as the growth rate remains constant. If the resulting exponential growth bothers you, more information must be added to the model.

Radioactive decay. If $c < 0$, $Ae^{ct} \to 0$ at $t \to +\infty$ although y never actually becomes 0. This is the expected behavior until you are down to your last atom.

Although e^t is the natural exponential, e is not a very intuitive number. For this reason, instead of specifying c, it is customary to describe decay in terms of a **half-life**, which is the value of t such that $e^{ct} = 1/2$.

The logistic model. The simplest way to avoid infinite growth is to **assume** that $dy/dt = ct(L - t)$, where L is some externally imposed **limit** to the size of the population. This leads to a growth model of the form

$$\frac{A}{1 + Be^{-kt}}.$$

The graph of this function is called the **logistic curve**.