The logarithm function. The exponential function with base b has been seen to be **monotonic** (increasing if $b > 1$, decreasing if $b < 1$). Such functions always have **inverse functions**. To use these inverses, it is only necessary to name them. The properties of the inverse are consequences of the properties of the original function.

The inverse of the exponential base b is the logarithm base b, denoted $\log_b x$ (for $b > 0$, $b \neq 1$). That is:

$$y = \log_b x \iff x = b^y.$$

Laws of logarithms. Here are the properties of the logarithm. Each is proved by relating it to a corresponding statement about exponentials:

$$\begin{aligned} \log_b(mn) &= \log_b(m) + \log_b(n) \\ \log_b m^n &= n \log_b m \\ \log_b 1 &= 0 \\ \log_b b &= 1 \end{aligned}$$

Natural logarithms. Exponentials with a special base b were considered **natural** because they had simpler properties in Calculus. The inverse function of

this exponential is called the **natural logarithm** and denoted ln. Thus,

$$y = \ln x \iff x = e^y.$$

Calculus of logarithms. The domain of a logarithm function consists only of **positive** values. Motivated by a desire to extend the domain without complicating formulas, the function

$$\ln |x|$$

can be considered. This is done in the text, but this extension can be easily added after the theory is complete, so the derivative will be given here only for $\ln x$ for $x > 0$.

If $y = \ln x$, dy/dx can be found using **implicit differentiation** of the equivalent expression $x = e^y$ to obtain

$$1 = (e^y) \cdot \frac{dy}{dx},$$

and solving to get

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Expressions that include natural logarithms can now be differentiated by using this formula as needed with the other properties of derivatives.

Other exponentials and logarithms. Let us go back to the general exponential function b^x. The inverse function relation between natural logarithms and exponentials gives

$$b = e^{\ln b}.$$

Thus,

$$b^x = \left(e^{\ln b}\right)^x.$$

The laws of exponents give

$$b^x = e^{(\ln b)x}$$

The chain rule then gives

$$\frac{d}{dx} b^x = e^{(\ln b)x} \frac{d}{dx}\big((\ln b)x \big) = b^x (\ln b).$$

The laws of exponents and logarithms allow an alternate description of $\log_b x$ that can be used for elemen-

tary purposes as well as for calculus:

$$\begin{aligned} y = \log_b x &\iff x = b^y \\ &\iff x = e^{(\ln b)y} \\ &\iff (\ln b)y = \ln x \\ &\iff y = \frac{\ln x}{\ln b}. \end{aligned}$$

Thus,

$$\frac{d}{dx}\big(\log_b x\big) = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{\ln b}\frac{1}{x}.$$

An example that must be included. Suppose $y = x^x$, what is dy/dx?

To differentiate, begin by writing

$$y = x^x = e^{x \ln x}.$$

Then

$$\begin{aligned} \frac{dy}{dx} &= e^{x \ln x} \frac{d}{dx}\big(x \ln x \big) \\ &= x^x \Big(x \frac{1}{x} + \ln x \Big) \\ &= \big(1 + \ln x \big) x^x. \end{aligned}$$

Logarithmic differentiation. A n easier variation on the same theme can be used if y is given as a product of several functions of x. Then $\ln y$ is a **sum** of the natural logarithms of those expressions. Implicit differentiation then gives $(1/y)(dy/dx)$ as the sum of the derivatives of those logarithms. Multiplying by y and writing it using the given expression as a function of x gives dy/dx entirely in terms of x.