The exponential function. If b is any **positive** real number, b^x can be defined algebraically for any rational x. Integer powers arise from repeated multiplication, and the case $x = 1/q$ is the q-th root function. Extraction of roots is the most familiar example of an **inverse function**: the definition is

$$y = x^{1/q} \iff x = y^q.$$

Since y^q with positive integer q is an **increasing function** of y on $[0, \infty)$ that is zero when $y = 0$ and becomes large as $y \to \infty$, it takes each positive real value at exactly point. Solving for y for given $x = y^q$ gives the q-th root.

The computation of roots is not as easy as we have made it seem, although it is well enough understood that it is a built-in function on calculator keyboards and a feature of (almost) all computer languages (the **Pascal** language forces the user to use an indirect description of powers and roots, but other languages allow us to pretend that raising to a power is a natural operation by providing a convenient symbol for this operation).

If $b > 1$, then b^x can be shown to be an increasing function of x (as far as we have defined it) with the property that values of x that are close give close values of b^x. This allows b^x, for arbitrary real x, to be approximated by b^x for rational x. This approach provides a definition that assures us that such a function b^x exists and has certain properties. Then we can use more advanced properties to give an efficient computation that can be made part of your calculator.

A Law of exponents. The idea of powers as **repeated multiplication** leads to the the property

$$b^x \cdot b^y = b^{x+y}. \qquad (E)$$

To obtain a rigorous verification of this property, start with the case of integer x and extend first to rational x and then to arbitrary real x using the definition of these extensions of the function b^x.

Note that this law requires that

$$b^x \cdot b^0 = b^{x+0} = b^x,$$

and this is only possible (since $b^x > 0$ for all x) if $b^0 = 1$. Then, using (E) with $x + y = 0$ tells us that b^{-x} is the **multiplicative inverse** of b^x, that is

$$b^{-x} = \frac{1}{b^x}.$$

Another law of exponents. Another appeal to the idea of powers as **repeated multiplication** leads to the the property

$$(b^x)^y = b^{xy}.$$

Again a proof that this holds in general requires stepping through the definition of the general exponential function.

An important special case is

$$(b^{-1})^x = b^{-x} = \frac{1}{b^x}.$$

If $0 < b < 1$, then $b^{-1} > 1$ and powers of b^{-1} can be expressed as powers of b. All exponential functions

can be described once one has exponential functions with $b > 1$.

A third property of exponents. If the base is a **product**, then a fixed power of this base will be a product of that power of the factors. In symbols,

$$(ab)^x = a^x \cdot b^x.$$

For integer powers, this follows from the commutative law of multiplication. The result for other exponents is built up step by step in the same way as the other rules.

Even for integer exponents, there is **no simple expression** for $(a + b)^x$. The **binomial theorem** gives a correct formula for positive integer exponents, but it is not the sort of formula that can be easily extended to other exponents.

The derivative of an exponential function. Let $f(x) = b^x$ and investigate the possibility of finding the derivative of $f(x)$. Since this is the first time that we have met this function, it is necessary to invoke the definition.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h \to 0} b^x \frac{b^h - 1}{h} \\ &= b^x \lim_{h \to 0} \frac{b^h - 1}{h} \end{aligned}$$

The limit in the last line, if it exists, is a **number** that depends on b. Since the graph of an exponential **appears to be smooth**, we expect that this limit does exist. The existence of the limit has been proved, so we can introduce

$$k(b) = \lim_{h \to 0} \frac{b^h - 1}{h},$$

and conclude that

$$f'(x) = k(b)b^x.$$

If $B = b^a$, then

$$\begin{aligned} k(B) &= \lim_{h \to 0} \frac{B^h - 1}{h} \\ &= \lim_{h \to 0} \frac{b^{ah} - 1}{h} \\ &= \lim_{h \to 0} a \frac{b^{ah} - 1}{ah} \\ &= ak(b) \end{aligned}$$

Thus, if one such limit exists, all will exist. This formula also shows that $k(b)$ takes all real values.

This course does not require complete proofs of the formulas of calculus, although fragments of proofs may be useful in recalling the formulas. We have shown enough to illustrate the relation between the laws of exponents and the formula for the derivative of an exponential function.

A special number. Let e denote the number such that $k(e) = 1$. Then

$$\frac{d}{dx}e^x = e^x.$$

We will learn more about the number e in the other sections of this chapter. The simple nature of this differentiation formula makes e the **natural base** for an exponential function.

Once we have learned how to differentiate **one** new function, we know how to differentiate anything obtained by combining it with other known functions using sums, products and composition of functions, since the rules of calculus were not specific to the functions previously used to illustrate them. There may be many ways to use these rules to differentiate a given function, but all expressions for the derivative must be equivalent.

Example: e^{2x} is both $e^{(2x)}$ and $\left(e^x\right)^2$.