Optimization. The relative extrema treated in previous sections are a technical invention. The chief reason for our interest in them is that they are **easy to find**. If you are going to invent a problem, you want to invent a problem that you can solve. It helps if it **looks important**, since that allows you to use your education to impress people.

Usually the real question is: "What is the largest that a certain quantity can be?" The quantity is often an estimate of a measurement based on some mathematical model. We have seen that these models are usually only valid on a limited domain, so this should be part of the analysis.

This leads to introducing the term **absolute maximum** of a function f on a domain D to represent a number M such that $f(x) \leq M$ for all $x \in D$ and $f(x) = M$ for some $x \in D$. **This asks for a lot!** It is possible that we have made so many requirements that they cannot be satisfied. For example, if D is the **open** interval $(0, 1)$, then the function $f(x) = x$ satisfies $f(x) < 1$ for all $x \in D$, but if $c < 1$, there will be $x \in D$ with $f(x) > c$. Although there is a clear

bound, the bound is not in the range of the function.

Fortunately, an absolute maximum can be shown to exist if f and D satisfy some **reasonable** conditions. The condition is that f be **continuous** and D be a **closed interval**.

Finding absolute maxima. Since the maximum is now an **attained** value of the function f, it is possible to modify the problem to one of finding a point x in the domain D such that $f(x) = M$.

While we are making reasonable assumptions, we can assume that f is not just continuous, but **differentiable** on D. The study of **relative maxima and minima** gave the following observation: if x is an **interior point** of D that is not a critical point, then $f(x)$ is not the largest (or the smallest) value of f on D.

Usually, this leaves only a finite number of points to be considered and assures us that the largest $f(x)$ in this finite set is the largest value of $f(x)$ is all of D.

The points that need to be considered are:

(1) the endpoints of the interval D;

(2) any critical points of f in D.

Simple examples. One of the simplest functions if a linear function $f(x) = mx + b$. In this case, $f'(x) = m$, so if $m \neq 0$, there are **no critical points**. Fortunately, we still have the **two** endpoints of D to consider, and one will give the minimum and the other will give the maximum.

Also consider $f(x) = 4 - (x+1)^2$ on all of $\mathbb{R}$. This isn't a closed interval, but **clearly**, the largest value is 4 and it is attained only when $x = -1$. If D were taken to be **any** closed interval containing -1, then the maximum would be at -1. Since $f'(x) = -2(x+1)$, this is a critical point of f. (If D is a closed interval that doesn't contain -1, f' will not change sign on D, so the function will be either everywhere increasing or everywhere decreasing on D and the extreme values will be taken at the endpoints.

Fine points of the theory. If f is not differentiable at a finite number of points of D, these points divide D into a few smaller intervals for which the general theorem applies. This tells us that the absolute extrema of f are taken on at one of the critical points of f **or** at one of the endpoints of the subintervals into which the theory required us to break D. These endpoints of subintervals include only the endpoints of D and all points at which $f'(x)$ does not exist.