**Graph Sketching**. The graphing calculator allows you to quickly view parts of the graph of any function you can describe. Much of what the calculator does is based on its ability to quickly perform arithmetic operations. This allows a large number of individual points to be plotted. You should **never** attempt to do this yourself. Indeed, tabulation a function at every integer between $-5$ and $5$ is a very bad approach to studying the function and its graph. It will take much less time to let your calculator find **hundreds** of points in this interval, and the calculator is much less likely to make a mistake.

After you have found a graph, the calculator allows you to modify the view to concentrate on different features of the graph.

One thing that **is** reasonable to do with hand calculation is to anticipate where these interesting features will be found. This section identifies such features and describes how they can be found. The examples will be **rational functions**, that is quotients of polynomials, because those are the only functions that we are currently able to differentiate. However, there are



enough such functions to show all the features we want to describe. Moreover, the methods are completely general and will apply to other functions.

**Vertical Asymptotes** The textbook uses the example

$$f(x) = \frac{x+1}{x-1}.$$

We already know that $x = 1$ is not in the domain of $f$ since $f$ is defined by a fraction that is zero when $x = 1$. Our study of limits distinguished two things that might happen at $x = 1$: (1) the numerator may also be zero at $x = 1$; (2) the numerator is bounded away from zero near $x = 1$.

When we were interested in limits, case (1) got most of the attention. This gave common factors of $x - 1$ in numerator and denominator, and removing those factors gave a **simpler** function that was equal to $f$ everywhere else. If the simpler function could be seen to be continuous, its value at $x = 1$ was the limit of $f(x)$ as $x \to 1$. The graph was drawn using the simpler function, but adding something to indicate that the one point where $x = 1$ should be omitted. Your calculator would probably only show a graph of the simpler function since they are equal **almost everywhere**.

Case (2) got brief mention earlier to justify the idea of extending the idea of limit to include infinite limits. The importance of infinite limits becomes clear when you consider a graph. If the denominator of a fraction is very close to zero while the numerator is bounded away from zero, then the quotient can be made arbitrarily large. A sample of function values are likely to include some large numbers. If these values are allowed to set the scale on the vertical axis, the graph will resemble a vertical line at the value where the function is undefined. Such a vertical line approached by a graph is called a **vertical asymptote**. For $f(x) = (x+1)/(x-1)$, there is a vertical asymptote at $x - 1$.

It is also possible to refine the study of vertical asymptotes by considering **one-sided limits**. For this $f(x)$, if $x > 1$ and close to 1, $x + 1 \approx 2$, while $x - 1 > 0$ but close to zero. This indicates not only that the quotient is unbounded, but also that it is a quotient of two positive numbers, and hence positive. Moving to the other side of the asymptote, $x + 1$ is still close to 2, but now the denominator is negative, so the quotient is negative. A graph appears to fall through the floor as $x$ (moving from left to right) approaches 1, and then to reappear on the ceiling a short distance on the other side of 1.



Vertical asymptotes are the graphical visualization of infinite limits.

**Horizontal asymptotes**. Section 2.4 also introduced the idea of **limits at infinity**. For rational functions, these could be found by expressing the function in terms of $1/x$ and using the fact that $1/x \to 0$ as $x \to \infty$ to employ standard results about limits. For our example, we have

$$f(x) = \frac{x+1}{x-1} = \frac{1+\frac{1}{x}}{1-\frac{1}{x}}.$$

If $x$ is **of large magnitude**, that is **far from zero**, $f(x)$ will be close to 1. A graph on a large interval will have most of the graph resembling the line $y = 1$. Such a line is called a **horizontal asymptote**.

**Other asymptotes**. When you have a **numerical** fraction, it is often rewritten as the sum of an integer and a fraction between 0 and 1. This is useful for recognizing the size of the number. The fractions between 0 and 1 are sometimes called **proper fractions**, but there are many **improper** fractions that have widespread use, like the approximations $22/7$ or $355/113$ for $\pi$. It would take longer to write these fractions if the integer part of 3 had to be identified separately.

There is also a concept of proper fraction for rational functions. A rational function is called proper if the **degree** of the numerator is smaller than the degree of the denominator. As in the case of numerical fractions, this is motivated by the operation of **long division**. If the numerator does not have smaller degree, you can do a division step, and if that term of the quotient is subtracted from the fraction, the result simplifies to have a quotient of lower degree. Every rational function can be written as the sum of a



polynomial and a proper rational function. For our original example,

$$f(x) = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}.$$

For $x$ of large magnitude, a proper rational function is close to zero, so the polynomial describes the behavior of the function. In the simple case with a denominator of degree 1, the numerator of a proper rational function will be constant. If that constant isn't zero, there will be a vertical asymptote when the denominator is zero.

For the function

$$x + 1 + \frac{2}{x}$$

whose graph appeared on the first exam (even though it was identified as a different function), the proper fraction in this expression will be small for $x$ of large magnitude, so the graph of the function will resemble the line $y = x + 1$. Although this use of long division is not difficult, it is rarely included in calculus textbooks, so you are not likely to be required to identify asymptotes that aren't either horizontal or vertical.

**Critical points**. When we functions were needed that failed to be continuous at certain points, or were continuous but not differentiable, we used a **definition by cases**. If a function is described by a single formula, it is likely to differentiable everywhere (unless it has a vertical asymptote). In the earlier sections of this chapter, we learned to look beyond this smoothness to identify interesting features.

First, we found $f'(x)$ and attempted to find where this was zero. This gave the **critical points** $x_i$ in the domain of $f$, which led to points $\left(x_i,\ f(x_i)\right)$ on the graph $y = f(x)$. On each segment between adjacent critical points, the graph is either increasing over the whole interval or decreasing over the whole interval. Most of these critical points will give **relative extrema** on the curve. The $y$ coordinates of such points can be used to determine a scale on the vertical axis.

Although the determination of $f'(x)$ helps gather in-



formation that is helpful in plotting $y = f(x)$, you should be careful not to leave $f'(x)$ lying around where features of **its** graph can get confused with the graph of $f(x)$.

If you are attempting to sketch a graph by hand, you can begin by plotting the points determined by the critical points and use knowledge of the sign of the derivative (or a second derivative test) to distinguish relative minima from relative maxima, allowing the graph near the critical point to be sketched. Connecting these fragments with smooth arcs should give a reasonable sketch. If you cannot complete the graph without introducing a new relative extremum, then a mistake has been made. It is necessary to correct the mistake before you can get a useful sketch.

**Points of inflection**. The values of $x$ where $f''(x) = 0$ determine candidates for points of inflection. You can plot the point $\left(x,\ f(x)\right)$ on the curve, and also evaluate $f'(x)$ to get the slope of the tangent line at the point. This allows the tangent line itself to be added to the sketch. If you also determine the sign of $f''$ between the points where $f''$ is zero, you know which side of these tangent lines contains the graph. Again, anything that makes it impossible to fit a curve to your constraints indicates a mistake.

Sketches produced by a calculator can be examined to show how well they illustrate the features identified by calculus.

**Exercises** Exercise 44 asks for a graph of

$$f(x) = \sqrt{x^2 + 5}$$

Here,

$$f'(x) = \frac{x}{\sqrt{x^2 + 5}}$$

which is zero only for $x = 0$, positive for $x > 0$ and negative for $x < 0$. The graph has a minimum at $x = 0$ with $f(0) = \sqrt{5} \approx 2.236$. Also,

$$f''(x) = 5(x^2 + 5)^{-3/2},$$

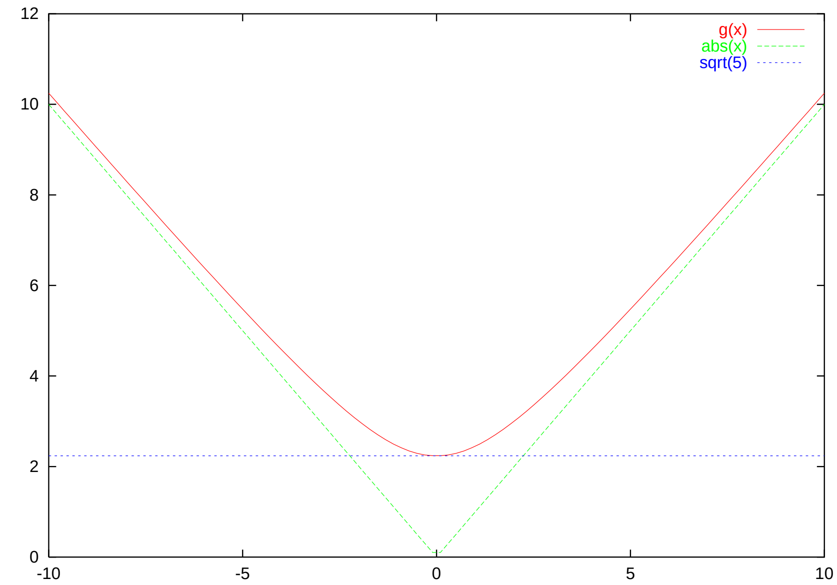
which is always positive, so the graph is everywhere concave upward. It is also possible to show that

$$\lim_{x \to \infty} \left( f(x) - |x| \right) = 0,$$



which shows that the graph is asymptotic to $y = |x|$.

Here is what a plotting program gives.




Exercise 57 asks for a graph of

$$g(t) = \frac{t + 1}{t^2 - 2t - 1}.$$

This has vertical asymptotes at the roots of the denominator, which are $1 \pm \sqrt{2}$, which are approximately 2.414 and $-0.414$. Since this is a **proper fraction**, the $t$ axis is a horizontal asymptote. Also $f(-1) = 0$, and this is the only value where $f(t) = 0$. (Note that the graph **is** allowed to cross an asymptote.) The function changes sign at this intercept and at the two vertical asymptotes. Then,

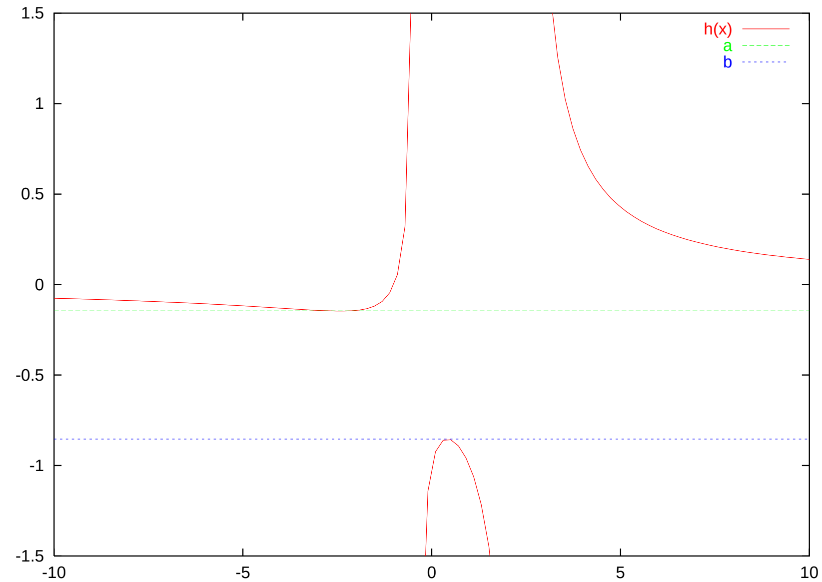
$$g'(t) = \frac{1 - 2t - t^2}{(t^2 - 2t - 1)^2}.$$

The denominator is a square, so the sign of $g'(t)$ is determined by the numerator. The numerator is zero at $-1 \pm \sqrt{2}$, which are just the negatives of the values giving the asymptotes, positive between these values and negative everywhere else. This gives a relative minimum at approximately $(-2.414, -0.146)$ and a relative maximum at approximately $(0.414, -0.854)$. Something more that can be used to check the graph is that each equation of the form $f(t) = c$ simplifies to a quadratic equation in $t$. Each horizontal line thus meets the graph in at most two points. For values of $c$ between $-0.854$ an $-0.146$, there are no values of $t$. In particular, this says that the **range** of $f$ consists of everything **outside** this interval. Up to this



point, we have not noticed any functions with such a complicated range. A closed form expression for the endpoints of the intervals in the range is $-(2 \pm \sqrt{2})/4$.

Here is a plot that shows the graph with its extreme values




We next turn attention to locating inflection points. The first step is to find

$$f''(t) = 2\frac{t^3 + 3t^2 - 3t + 3}{(t^2 - 2t - 1)^3}$$

The numerator is seen (for example, by graphing **it**) to have only one real root, approximately $-3.951$. This gives an inflection point at $(-3.951, -0.131)$ (approximately), with a tangent line of slope approximately $-0.013$. The graph is concave downward to the left of this point and between the asymptotes, concave upward elsewhere. A graph showing the tangent line at the inflection point is