Old business. There was a request in the last lecture to show some examples of the quotient rule, so that is how we will begin.

New business. There isn't much that needs to be said about Section 3.5. Once you accept that $f'(x)$ is a function, then you can ask about differentiating it. For the moment, computing second derivatives requires only having an expression for the first derivative. The only real concern is a notation for higher order derivatives.

If derivatives are denoted by writing a **prime** after the name of a function, then successive derivatives are denoted by writing more primes. When you tire of writing primes, you count the number that you need to write and enclose that number in parentheses as a superscript. Thus, the third derivative of $f(x)$ is likely to be denoted $f'''(x)$, but the fifth derivative would be $f^{(5)}(x)$.

If derivatives are denoted by writing D in front of the quantity to be differentiated, then an n-th derivative of a function $f(x)$ is denoted $D^n f(x)$.

If y is an **expression** containing x, and the function g is such that $g(x)$ is equal to the expression y, then one uses the notation

$$g'(x) = \frac{dy}{dx}.$$

In this notation, the role of D is played by

$$\frac{d}{dx},$$

so a second derivative of y with respect to x is denoted

$$\left(\frac{d}{dx}\right)^2 y \text{ or } \frac{d^2y}{dx^2}.$$

Higher derivatives follow the same pattern. The textbook also allows y' as a synonym for dy/dx, but I prefer to use primes **only** with functions and d/dx only with expressions.

A few examples should make this clear.

Just business: Cost. Section 3.4 describes some of the terminology used in applying calculus to economics.

The **cost function** $C(x)$ represents the cost of producing x units of a certain product. In some applications, a fixed period of time is specified and $C(x)$ represents the expenses of producing x units in that fixed period of time.

If x units have been produced, the cost of producing **one more**, is $C(x+1) - C(x)$, which really is

$$\frac{C(x+1) - C(x)}{(x+1) - (x)}. \qquad (*)$$

In other words, it is a **difference quotient**. In the original discussion of derivatives, we took difference quotients between **arbitrarily close** values of x so that we could obtain a derivative. Each actual difference quotient was considered an approximation to the derivative.

Once we **have** the derivative, we realize that derivatives are **easy to compute**, so it makes sense to treat

the derivative as an approximation to a difference quotient. In particular, if we are working with a model in which x takes only integer values, the smallest difference between vales of x is 1, so the difference quotient in $(*)$ can be approximated by $C'(x)$. With this interpretation, $C'(x)$ is called the **marginal cost**. It is one way to describe the cost of **one** item.

Alternatively, $C(x)/x$ gives the individual cost of a representative of a set of x items made. This is called the **average cost** of each item when x items are made.

Revenue. If the items are sold, they bring in some money. The fancy word for this is **Revenue**. If x items are sold at one time, each will have a **price** $p(x)$, and the revenue $R(x)$ will satisfy

$$R(x) = xp(x).$$

This equation may also be interpreted as saying the price is average revenue.

There is also a **marginal** revenue given by $R'(x)$.

Typically, the price at which an item can be sold is a decreasing function of the number of items available, so $p'(x) < 0$.

The revenue function $R(x)$ is the product of the increasing function x with the decreasing function $p(x)$. It is common for such a function to increase to a maximum value and then decrease. This leads to there being a value of x for which $R(x)$ has a maximum value.

Profit. Usually a portion of the revenue is used to pay the manufacturing cost. What is left over is called **Profit** and denoted $P(x)$. Thus,

$$P(x) = R(x) - C(x).$$

Usually, it is the profit function that one seeks to maximize.

Inverse functions. Since $p(x)$ is a decreasing function, there will be only one x corresponding to a given p. If market research leads to some knowledge of $p(x)$ without actually experimenting with producing different numbers of items, it also predicts a function $x = f(p)$ that gives the number x of items that will be sold at price p.

Elasticity of demand. The **elasticity function** compares the **relative** change in x to the relative change

in price. The definition includes an arbitrary change of sign to assure that this measurement will usually be positive.

The definition is

$$E(p) = -\frac{pf'(p)}{f(p)}.$$

Writing $x = f(p)$, the alternative notation is

$$E(p) = -\frac{p}{x}\frac{dx}{dp}.$$

If $E(p) > 1$, one says "demand is elastic".

What does it mean? Using the notation of expressions, rather than functions, we have $R = px$. Treating p as our independent variable,

$$\frac{dR}{dp} = x + p\frac{dx}{dp} = x - Ex = x(1 - E).$$

This shows that elastic demand corresponds to the region where R is a decreasing function of p. That is, when demand is elastic, a decrease in price creates sufficient new demand that there is an increase in revenue. In a sense, the market works for the benefit of everyone.

Without Calculus, the description of the properties of E would be a long story, and the effect of changes in p on R would probably be described in terms of a sequence of steps involving one change to lead to another. In the model using calculus, every change has an instant effect on all related quantities. While the real world may show more inertia, it seems reasonable that formulas based on using derivatives to approximate difference quotients will accurately reflect market forces and describe trends after transient effects have run their course.