Section 6.3 This section deals with the definition of the integral as a limit of a sum. When the terms of the sum are all positive, the sum is viewed as an approximation to the area under the graph of a function. The exercises explore several aspects of this definition.

Exercise 1 A picture is given that shows the graph of a function (even though no formula is given, we can **see** that it is a function because vertical lines meet the graph in only one point) between $x = 1$ and $x = 3$. The domain is divided into 6 subintervals and rectangles are drawn over each subinterval with height equal to the value of the function at the **midpoint** of the interval. Each of these subintervals has width

$$\Delta x = \frac{3-1}{6} = \frac{1}{3}.$$

Forming a **table** of the information shown on the graph:

- on $[3/3, 4/3]$, use $f(7/6) = 1.9$;
- on $[4/3, 5/3]$, use $f(9/6) = 1.5$;
- on $[5/3, 6/3]$, use $f(11/6) = 1.8$;
- on $[6/3, 7/3]$, use $f(13/6) = 2.4$;
- on $[7/3, 8/3]$, use $f(15/6) = 2.7$;
- on $[8/3, 9/3]$, use $f(17/6) = 2.5$.

The **Riemann sum** for this choice of points x_i is the product of Δx and the sum of the $f(x_i)$ tabulated above. Thus,

$$\text{Sum} = (1.9 + 1.5 + 1.8 + 2.4 + 2.7 + 2.5)\frac{1}{3} = 4.27\,.$$

Since the only restriction on the x_i is that x_i lie in the i^{th} subinterval, Riemann sums for this partition can take **visibly different** values. In addition to being the only choice allowed by the limited explicitly given data, the choice of the **midpoint** of each interval for x_i has other benefits. The best that can be said about the accuracy of a general Riemann sum (for nice enough f) is that the difference between a Riemann sum with n intervals and the integral that is the limit of such sums is bounded by a **constant multiple** of $1/n$. However, if f has a **continuous second derivative**, the systematic use of the midpoint for x_i gives sums for which this difference is bounded by a constant multiple of $1/n^2$.

Better ways of approximating integrals are known. Your calculator claims to be able to compute numerical values of definite integral. The method used is **Simpson's Rule**, which uses a **weighted average** of values of the function at points in each interval instead of a single $f(x_i)$. For functions with **continuous fourth derivatives**, the difference is a bounded multiple of $1/n^4$. There are some special cases in which the failure of the **smoothness** hypothesis (i.e. the assumption that certain derivatives exist and are continuous) can be important. A simple example is

$$\int_0^1 x^{1/2}\,dx\,.$$

Although it **appears** possible to differentiate the function $f(x) = x^{1/2}$ as many times as you like, $f'(x)$ is not defined at $x = 0$, and the bounds require that certain derivatives be continuous on the **closed** interval of integration. Riemann sums still converge for this integral but more slowly.

Exercise 3 The region under the graph of $y = 3x$ for $0 \le x \le 2$ is a **triangle**, whose base has length 2 and whose height is 6. Thus,

$$\text{Area} = \frac{1}{2}(2)(6) = 6.$$

A Riemann sum with 4 subintervals with each x_i taken to be the **left endpoint** of the subinterval has

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

is based on the table:

- on $[0/2, 1/2]$, use $f(0/2) = 0/2$;
- on $[1/2, 2/2]$, use $f(1/2) = 3/2$;
- on $[2/2, 3/2]$, use $f(2/2) = 6/2$;
- on $[3/2, 4/2]$, use $f(3/2) = 9/2$;

Thus,

$$\text{Sum} = \left(\frac{0}{2} + \frac{3}{2} + \frac{6}{2} + \frac{9}{2}\right)\frac{1}{2} = \frac{9}{2}.$$

Since the **exact** Area is known, we see that this sum is $3/2$ too small.

Repeating this with 8 subintervals, again using a Riemann sum with each x_i taken to be the **left endpoint** of the subinterval gives

$$\Delta x = \frac{2-0}{8} = \frac{1}{4}$$

is based on the table:

- on $[0/4, 1/4]$, use $f(0/4) = 0/4$;
- on $[1/4, 2/4]$, use $f(1/4) = 3/4$;
- on $[2/4, 3/4]$, use $f(2/4) = 6/4$;
- on $[3/4, 4/4]$, use $f(3/4) = 9/4$;
- on $[4/4, 4/4]$, use $f(4/4) = 12/4$;
- on $[5/4, 6/4]$, use $f(5/4) = 15/4$;
- on $[6/4, 7/4]$, use $f(6/4) = 18/4$;
- on $[7/4, 8/4]$, use $f(7/4) = 21/4$;

Thus,

$$\text{Sum} = \left(\frac{0}{4} + \frac{3}{4} + \frac{6}{4} + \frac{9}{4} + \frac{12}{4} + \frac{15}{4} + \frac{18}{4} + \frac{21}{4}\right)\frac{1}{4} = \frac{21}{4}.$$

Now, we see that this sum is $3/4$ too small.

It can be shown that doubling the number of points while continuing to take the left endpoint in this Riemann sum **always** divides the difference from the true area in half. Also taking the **right endpoint** gives a result that overestimates the area by exactly as much as the use of the left endpoint underestimates it.

Since the function is linear, **averaging** the values at the endpoints gives the same result as **evaluating** at the midpoint. In this case, that gives the exact area for any partition of the interval.