Section 6.1

Exercise 1 In this exercise, you are asked to verify that

$$F(x) = \frac{1}{3}x^3 + 2x^2 - x + 2$$

is an antiderivative of

$$f(x) = x^2 + 4x - 1.$$

This could never be an **exam question**, since it is almost impossible to leave a written record that explains how these instructions were followed. However, it is useful as a **homework exercise**, since it reminds you that it is possible to check answers. The **verification** consists of finding $F'(x)$ by differentiating term-by-term and **noticing** that the result is f(x).

I noticed — did you?

Exercise 8 Here we are to first verify that

$$G(x) = e^x$$

is an antiderivative of

$$g(x) = e^x.$$

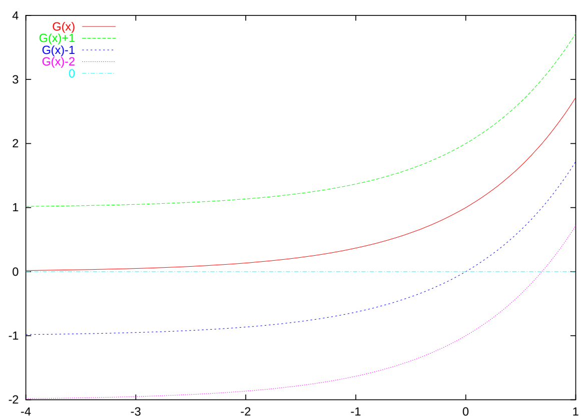
Again, we know how to differentiate to find G, and we **notice** that the result is equal to the given g.

Then, we are asked to **find all antiderivatives** of g. This section told us that this means that we should add an **arbitrary constant** to G to obtain something that we **write as**

$$e^x + C.$$

This seems too easy. Of course, **it is**, but some problems will give an additional condition that requires us to replace C by a **particular** number found by solving for C the equation expressing the condition that an antiderivative satisfies the condition. Having an expression for the general antiderivative allows us to begin the second part of such a problem

Another use of C is to specialize it to allow several different antiderivatives to be graphed. Here, we graph $G(x)$, $G(x)+1$, $G(x)-1$, and $G(x)-2$.



Exercise 10 This asks for the **indefinite integral**

$$\int \sqrt{2}\,dx = \sqrt{2}\,x + C.$$

Here, $\sqrt{2}$ is constant so its antiderivative is based on

$$\frac{d}{dx}x = 1.$$

Exercise 15 This asks for the **indefinite integral**

$$\int x^{2/3}\,dx = \frac{3}{5}x^{5/3} + C.$$

We illustrate at process combining **discovery** and **verification** in which we start from simple differentiation formulas **suggested by general principles**, and modify them until we find the correct integral. Since the course has emphasized differentiation, this is an alternative to introducing a second collection of formulas for integrals. The formulas for these two **inverse** operations are sufficiently similar that they could be easily confused. Since differentiation formulas are based on a definition as a limit whose steps can be retraced when necessary, this is the more reliable method for remembering the formulas. Common practice (including the formula sheet for this course) is just the opposite: the assumption there is that you already know the differentiation formulas but will need to consult a table for the inverse operation. Quite elaborate tables have been prepared that summarize all the tricks that have been used in integration, but the problems you are likely to meet in this course are simple enough that you only need to know how to **verify** that you have found the clue that leads to the correct answer. The notation for our solution will consist of a **table** of $F : F'$ pairs. For this problem, it looks like

$$\begin{aligned} F &: F' \\ x^n &: nx^{n-1} \\ x^{5/3} &: \frac{5}{3}x^{2/3} \\ ax^{5/3} &: \frac{5}{3}ax^{2/3} \\ \frac{3}{5}x^{5/3} &: x^{2/3} \end{aligned}$$

Beginning with the general power rule, we see the F must contain $x^{5/3}$ if F' is to contain $x^{2/3}$. Then a constant multiple of a function has a derivative that is the same constant multiple of the derivative of that function. Now, a little algebra (**very little**) shows that we can make the coefficient $(5/3)a$ equal to the given coefficient of 1 by setting $a = 3/5$.

The last line of the table **verifies** one antiderivative, and the $+C$ gives a general formula for all.

A Variant This **tabular method** comes into its own when you can see the form of the solution, but there will be several **chain rule** factors that get combined when **verifying** the answer. The next section will introduce a **method of substitution** that successfully inverts the chain rule, but it requires that the details of the factors be introduced in different parts of the process and combined at the end. This often introduces errors, so it should always be checked. It may be more accurate to use the general method to find the **shape** of the solution and save the details for a verification step. Here is an example.

$$\int (2x+1)^{4/3}\,dx$$

We expect that, up to a constant factor, one answer will be $(2x+1)$ raised to a power that is one more than the given $4/3$, so we start there.

$$\begin{aligned} F &: F' \\ (2x+1)^{7/3} &: \frac{7}{3}(2x+1)^{4/3}(2) \\ &: \frac{14}{3}(2x+1)^{4/3} \\ a(2x+1)^{7/3} &: \frac{14}{3}a(2x+1)^{4/3} \end{aligned}$$

To get the given coefficient of 1, we need $a = 3/14$, so the answer is

$$\int (2x+1)^{4/3}\,dx = \frac{3}{14}(2x+1)^{7/3}$$

Exercise 17 Find

$$\int x^{-5/4}\,dx.$$

After the initial power rule observation, we have

$$\begin{aligned} F &: F' \\ x^{-1/4} &: \frac{-1}{4}x^{-5/4} \\ ax^{-1/4} &: \frac{-a}{4}x^{-5/4} \end{aligned}$$

Solving $-a/4 = 1$ gives $a = -4$. (Turning the problem into algebra avoids a formula for the answer that asks to divide by $-1/4$. This is one place where algebra is clearer that arithmetic.) Thus,

$$\int x^{-5/4}\,dx = -4x^{-1/4} + C.$$

Exercise 20 Find

$$\int \frac{1}{3x^5}\,dx.$$

This should be rewritten as

$$\int \frac{1}{3}x^{-5}\,dx$$

to isolate the constant factor and to use negative exponents to avoid have x in a denominator. We are now in a position to use a power rule.

$$\begin{aligned} F &: F' \\ ax^{-4} &: -4ax^{-5} \end{aligned}$$

and we need $-4a = 1/3$, so$a = -1/12$. Now that the calculus has been done successfully, we can put the x back in the denominator to get

$$\int \frac{1}{3x^5}\,dx = \frac{1}{12x^4} + C.$$

Exercise 25 Sometimes, several terms need to be combined by the **sum rule**. The table will be used to find the terms (we will identify them with (•) at the end of the line). Find

$$\int x^2 + x + x^{-3}\,dx.$$

We build a table omitting the general statement of the power rule and explicit identification of the coefficient a used in setting up the easy algebra to find the correct coefficient. With practice, this approach becomes efficient by limiting what needs to be written. You should add as many details as you need to make the process clear.

$$\begin{aligned} F &: F' \\ x^3 &: 3x^2 \\ \frac{1}{3}x^3 &: x^2 \qquad (\bullet) \\ x^2 &: 2x \\ \frac{1}{2}x^2 &: x \qquad (\bullet) \\ x^{-2} &: -2x^{-3} \\ -\frac{1}{2}x^{-2} &: x^{-3} \qquad (\bullet) \end{aligned}$$

Combining these terms and adding an arbitrary constant gives (with exponents written in decreasing order)

$$\int x^2 + x + x^{-3}\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C - \frac{1}{2}x^{-2}.$$

Exercise 29 Find

$$\int 1 + x + e^x\,dx.$$

Here is the table.

$$\begin{aligned} F &: F' \\ x &: 1 \qquad (\bullet) \\ x^2 &: 2x \\ \frac{1}{2}x^2 &: x \qquad (\bullet) \\ e^x &: e^x \qquad (\bullet) \end{aligned}$$

Hence,

$$\int 1 + x + e^x\,dx = x + \frac{1}{2}x^2 + e^x + C.$$

Exercise 37 It is not always necessary to use x an the **independent variable**. The notation reminds us of the variable by including it in the **differential** that marks the end of the notation for an integral. Find

$$\int \frac{u^3 + 2u^2 - u}{3u}\,du.$$

It is important to **use algebra first** to put this in the form that derivatives first appear. In this case, the denominator needs to be divided into **each term** of the numerator and numerical coefficients isolate from the powers of x. The result is

$$\int \frac{1}{3}u^2 + \frac{2}{3}u - \frac{1}{3}\,du.$$

The answer is now seen to be

$$\frac{1}{9}u^3 + \frac{1}{3}u^2 - \frac{1}{3}u + C.$$

Exercise 42. Another example, this time with fractional exponents and a different **independent variable**. Find

$$\int \sqrt{t}\left(t^2 + t - 1\right)dt.$$

Perform the multiplication to get

$$\int t^{5/2} + t^{3/2} - t^{1/2}\,dt.$$

Also note that **fractional exponents** are used in place of radicals since that is the form preferred in calculus. Now, we can work term-by-term to get

$$\frac{2}{7}t^{7/2} + \frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2} + C.$$

A Final Exercise A more striking example of the need to do algebra first is

$$\int \left(t+1\right)\left(t+3\right)dt$$

$$\int t^2 + 4t + 3\,dt$$

$$\frac{1}{3}t^3 + 2t^2 + 3t + C$$