Section 4.2 Applications of the second derivative

Exercises 2, 4, and 6 were based on pictures in the textbook that did not include equations of the functions being graphed. The demonstration in lecture consisted of a description of the identification of the requested features on the graph. This survey can only approximate that demonstration.

Exercise 10 asked to select the graph with certain properties. Reference to the text is required there also.

Exercise 26 was also discussed. This can be transcribed in more detail.

Exercise 2 The x axis has tick marks only at 1 and 2 although it appears to continue past 3. In this section, we concentrate on identifying concavity, but the increasing and decreasing portions of the curve will be useful for identifying portions of the curve. The graph is **increasing** (so $f'(x) > 0$) at $x = 0$ and has a **relative maximum** (with $f'(x) = 0$) near $x = 1$. Thus $f'(x)$ is **decreasing** over this interval. In fact, it is **steadily decreasing**, and continues to be steadily decreasing until about $x = 4/3$, indicating the curve is **concave downward** on this interval. The slope of the tangent line has reached a (relative) minimum, after which it increases. A relative extremum of $f'(x)$ is one way to identify an **inflection point** on the graph of $f(x)$. At this inflection point $f'(x) < 0$, so the graph is decreasing. The function $f(x)$ has relative minimum near $x = 2$, and the derivative has increased from its negative value at the inflection point to zero, and it continues to increase to positive values in the increasing portion of the curve at the right of the picture. Again, $f'(x)$ is **steadily** increasing, indicating that the portion shown to the right of the inflection point is **concave upward**.

Exercise 4 The graph splits into **three separate arcs** because $f(x) \to \pm\infty$ as x approaches some values that appear to be ± 4. (In section 4.3, we will learn that the vertical lines added to the background of the picture to separate the arcs are called **vertical asymptotes**.) The outer portions of the figure are concave upwards and the inner portion is concave downwards. Note that the concavity of this graph changes at the asymptotes although $f(x)$ is increasing for **all** negative x in the domain of f.

Exercise 6 This curve appears to have been defined by cases because of the **sharp corners** at $x = 1$ and $x = 5$. A reasonable guess of the defining function for this graph is

$$\left|(x-3)^2 - 4\right| = \begin{cases} 4 - (x-3)^2 & \text{if } 1 < x < 5 \\ (x-3)^2 - 4 & \text{otherwise} \end{cases}.$$

The picture shows a decreasing function (so $f'(x) < 0$) at $x = 0$, that becomes more **gently** decreasing ($f'(x)$ less negative) as x moves from 0 to 1. Again, this happens **steadily**, so $f'(x)$ is increasing and the graph of $f(x)$ is **concave upward**.

At $x = 1$,

$$\lim_{x \to 1^-} f'(x) < 0 < \lim_{x \to 1^+} f'(x)$$

so there is no derivative at this point. The positive derivative a little to the right of $x = 1$ decreases to zero at the relative maximum (at $x = 3$ in the formula above) and continues to become more negative up to the corner at $x = 5$. This decreasing nature of $f'(x)$ indicates that the graph of $f(x)$is **concave downward** and $f''(x) < 0$ on the interval from 1 to 5.

At $x = 5$, there is an abrupt change to a positive derivative, which **continues to increase** for larger x. The right side of the curve is again **concave upward**.

Exercise 10 Three graphs are shown in which the point $(1, 2)$ on the graph of $f(x)$ is **emphasized**. In all cases, $f(x)$ is everywhere increasing, so $f'(x) > 0$ **where it is defined**, which includes all x except for $x = 1$. However, graphs (a) and (c) **do not have values** of $f'(1)$. In (a), there is a **vertical tangent**, signifying that $f'(x)$ is unbounded near $x = 1$. In (c), there is **corner** at $(1, 2)$. Since $f'(1)$ does not exist, it is not possible differentiate $f'(x)$ to obtain $f''(x)$ at $x = 1$. Only (b) has a chance of having $f''(1) = 0$. The picture shows that this $f(x)$ is concave downward for $x < 1$ and concave upward for $x > 1$, so if $f''(1)$ is defined, the only possible value is zero.

Note that (a) also shows a change of concavity, with $f(x)$ concave upward for $x < 1$ and concave downward for $x > 1$. However, this $f(x)$ appears to have a very large second derivative for x near 1.

Exercise 26 We have the formula

$$f(x) = 3x^4 - 6x^3 + x - 8,$$

from which we find

$$f'(x) = 12x^3 - 18x^2 + 1$$

and

$$f''(x) = 36x^2 - 36x = 36x(x - 1).$$

Thus, $f''(x) < 0$ for $0 < x < 1$. The graph of $f(x)$ is **concave downward** only on this interval. To get an appropriate **window** for this portion of the graph, note that $f(0) = -8$, $f'(0) = 1$, $f(1) = -10$, and $f'(1) = -5$. The lines $y = x - 8$ and $y = -5(x - 1) - 10$ are the tangent lines at the inflection points at $x = 0$ and $x = 1$ (respectively). Since the graph is concave downward,l it lies below both of these tangent lines. The lines meet at $(1/2, -15/2)$, so this portion of the curve cannot have values of $f(x)$ greater than $-15/2$.

The cubic equation $f'(x) = 0$ has three real roots, but they are **not** rational, so there is no simple expression for the roots of this polynomial, which determine the **critical values** of $f(x)$. There is a relative maximum somewhere between 0 and 1, and relative minima in each of regions with $x < 0$ or $x > 1$, which can be located as accurately as desired even if there isn't a simple formula for their values. The interval $[-1, 2]$ is already big enough to contain all critical points and almost big enough to contain all real roots of $f(x) = 0$.

By considering intersections of tangent lines at other points, we find that a graph for $-1 \le x \le 2.5$ and $-14 \le y \le 1$ will contain all interesting features of this graph, including the intercept at $x = -1$. Here is a graph of this functions with some tangent lines added to show how they frame the curve in regions where the second derivative has a fixed sign.