The Bieberbach Conjecture

*A famous unsolved problem and the
story of de Branges' surprising proof.*

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Introduction

The biggest mathematical news of 1984 was the proof of the 68-year-old Bieberbach conjecture. Louis de Branges of Purdue University had solved what Felix Browder of the University of Chicago described as "one of the most celebrated conjectures in classical analysis, one that has stood as a challenge to mathematicians for a very long time." Beginning in spring 1984, the news spread quickly through the mathematical world as de Branges lectured in Europe and as preprints and informal communications circulated. Even the New York Times (Sept. 4, 1984, p. C12) reported the story—incorrectly. What was all the fuss about?

The details of de Branges' ingenious proof are well beyond our scope. We will focus instead (mainly) on the conjecture itself. What does it say? Why would one conjecture it? What partial results are "obvious"? Why did so many mathematicians work so hard at the problem for so long? Who contributed to its solution? What comes next?

The Bieberbach conjecture is an attractive problem partly because it is easy to state—it says that under reasonable restrictions the coefficients of a power series are not too large. More precisely, let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

be a power series in z, with complex coefficients $a_2, a_3, \ldots$. Assume that $f(z)$ converges for all complex numbers z $(= x + iy)$ with $|z|$ $(= (x^2 + y^2)^{1/2}) < 1$, and that the function $f(z)$ is one-to-one on the set of such z. Then

BIEBERBACH CONJECTURE. $|a_n| \leqslant n, n = 2, 3, \ldots$. *The inequality is strict for every n unless f is a "rotation" of the* **Koebe function**

$$k(z) = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots .$$

(The term "rotation" is defined at the end of the next section.)

The Bieberbach conjecture (hereafter, "BC") is at heart an assertion about extremality of the Koebe function. Therefore, understanding the Koebe function and why it is the natural candidate to be "biggest" in the sense of the conjecture is a recurring theme in this essay.

The BC first appeared in a footnote to a 1916 paper [4] of the German mathematician Ludwig Bieberbach, of which the principal result was the *second* coefficient theorem: $|a_2| \leqslant 2$; equality

Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln.

Von Prof. Dr. Ludwig Bieberbach
in Frankfurt a. M.

(Vorgelegt von Hrn. Frobenius am 6. Juli 1916 [s. oben S. 775].)

The article

¹ Daß $k_n \geq n$ zeigt das Beispiel $\sum n z^n$. Vielleicht ist überhaupt $k_n = n$.

and the footnote that led to it all.

holds essentially only for the Koebe function. As we shall see, consequences of Bieberbach's theorem are at least as important as the theorem itself for understanding the BC, and in univalent function theory generally. Bieberbach himself proved no other analogous coefficient theorems. Before de Branges' general proof, $|a_n| \leq n$ was known only for $n \leq 6$.

The BC's words are familiar from elementary *real* calculus, but the meaning and interest of the conjecture are essentially rooted in *complex* analysis. This article aims to provide some mathematical and historical context for the BC. It is neither an exhaustive summary of the gigantic body of research in univalent function theory (which comprises thousands of papers; see, e.g., [2]) nor a careful presentation of de Branges' solution. It is an eclectic sample of background material, related results, and exercises related either to the conjecture itself or to standard ideas and techniques of the subject. We are less concerned with de Branges' dramatic achievement itself than with the stage setting—lighting, scenery, and backdrop—against which it is played.

Why write this essay when a wealth of clear and inviting expository books and articles already exists? (See, e.g., [3], [7], [8], [9], [11], [14], [15].) This article, which might have been titled "BC for Tourists" had not A. Baernstein [3] already used that phrase, is an invitation and a preamble to these sources, which assume more mastery of complex analysis. We assume that the reader is familiar with the rudiments of that subject (as developed in any undergraduate text), but *only* the rudiments; most of what is used is stated explicitly. Because all of the cited works contain extensive bibliographies, references for standard theorems discussed here are usually omitted.

What does the BC say?

The adjective "analytic" applied to a complex-valued function of a complex variable means "continuously differentiable in the complex sense"; i.e., f is a function of z, defined on a domain D in $\mathbb{C}$, and the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists and is a continuous function of z_0. This generalization from real-variable calculus looks innocuous, but it is not. Analytic functions have pleasant properties not shared by their real-differentiable cousins. Most important, every analytic function can be expanded in a (complex) Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

about any point z_0 of its domain; the series converges to $f(z)$ in any circular neighborhood centered at z_0 that lies within the domain D. (Conversely, any series $\sum a_n z^n$ that converges for all z in D defines a function analytic on D.) None of this is true of an arbitrary real differentiable function: f may be no more than once differentiable, and so have no Taylor series at all; f's Taylor series, if it exists, may diverge; f's Taylor series, if it converges, may converge to a limit other than f.

Thus, an analytic function defined on any domain is, at least *locally*, a power series. An analytic function $f(z)$ that happens to be defined on the **unit disk** $\{z: |z| < 1\}$ in $\mathbb{C}$—hereafter referred to as D—is *globally* a (convergent) power series $\sum_{n=0}^{\infty} a_n z^n$. The sequence $\{a_0, a_1, a_2, \dots\}$ of coefficients completely determines f's behavior. How, then, are analytic and geometric properties of f (e.g., univalence, boundedness, convexity) reflected in properties of f's coefficients (e.g., growth rate, individual bounds)? The BC (now de Branges' theorem) is the best-known "coefficient problem" of this kind.

The BC concerns functions which are analytic and also **univalent** on the unit disk. "Univalent" is the complex analyst's term for "one-to-one": $f(z_1) \neq f(z_2)$ unless $z_1 = z_2$. Synonyms for "analytic univalent" include the German **schlicht** (simple, unpretentious) and the Russian **odno-listni** (single-sheeted). These words emphasize a geometric property of a univalent function $w = f(z)$: it maps the disk D in the z-plane one-to-one and onto a domain $f(D)$ in the w-plane. (See FIGURE 1.) By contrast, the *p-valent* function $f(z) = z^p$ maps D onto D, but each image
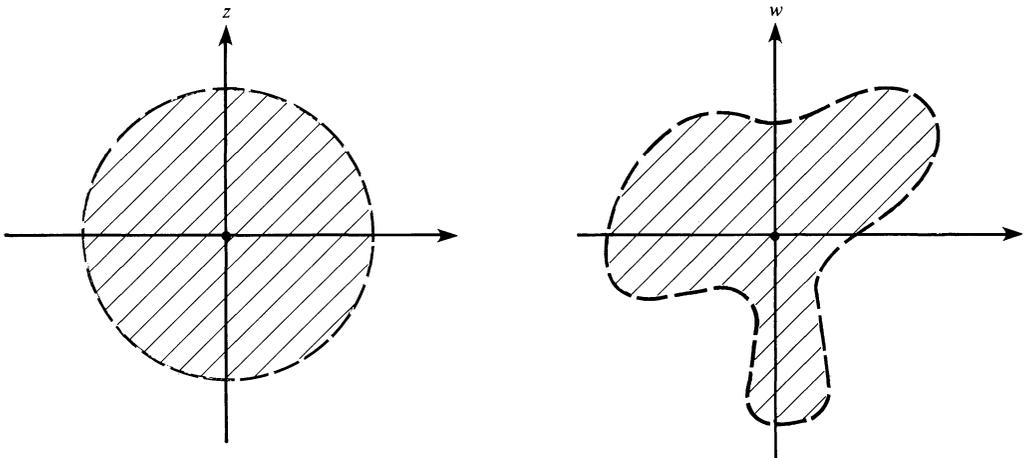
FIGURE 1

point (except $w = 0$) has p different preimages. More picturesquely, $f(z) = z^p$ can be viewed as mapping D in the z-plane *univalently* onto a spiral-like surface with p layers ("sheets") hovering above D in the w-plane.

The most important property of univalent analytic functions is the famous **Riemann mapping theorem**, stated in 1851: every proper subdomain of the complex plane that is simply connected (without "holes") is the image of the unit disk under a univalent analytic mapping $f(z)$. The mapping function $f(z)$ is uniquely determined by the domain D, the image point $f(0)$ in D, and the requirement that $f'(0)$ be a positive real number. Thinking of univalent analytic functions as "Riemann mappings," one naturally wonders how analytic properties of f and geometric properties of the image domain $f(D)$ reflect each other. This is the viewpoint of *geometric function theory*; until de Branges' proof, the BC was its main problem.

The BC, as usually stated, is an assertion about a special family of analytic functions on D:

DEFINITION. The **normalized schlicht class**, denoted S, is the family of univalent analytic functions $f : D \to \mathbb{C}$ for which
  1. $f(0) = 0$
  2. $f'(0) = 1$.
Conditions 1 and 2 say that an S-function has power series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n.$$

The normalizing assumptions simplify the BC's appearance by eliminating irrelevant constants. Fortunately, the normalizations are harmless: if f is *any* univalent analytic function on D, then $g(z) = (f(z) - f(0))/f'(0)$ is in S, and properties of f are easily deduced from those of g. Geometrically, studying g rather than f corresponds to first *translating* the image domain by the vector $f(0)$, *dilating* by the factor $|f'(0)|$, and *rotating* through the angle $\arg(f'(0))$. All of these operations are reversible.

The identity $f(z) = z$ is in one sense a prototype for S: the normalization means that every S-function agrees with the identity up to order one at the origin. The most important nontrivial schlicht function is the Koebe function $k(z) = z/(1 - z)^2$, named for the German mathematician P. Koebe, whose achievements include the first correct proof the Riemann mapping theorem. Using the fact that

$$k(z) = z \frac{d}{dz}\left[\frac{1}{1 - z} \right],$$

it is easy to see that k can be written as the power series

$$k(z) = z + 2z^2 + 3z^3 + 4z^4 + \cdots,$$

which converges for every z in D. Bieberbach conjectured that k is at the other extreme from the identity function—*every* coefficient of k is as large as possible. Why did Bieberbach guess this almost 70 years before it was proved? As a first step toward understanding k, let's check explicitly that it is univalent on D and find its image. The identity

$$k(z) = \frac{1}{4}\left[\left(\frac{1 + z}{1 - z} \right)^2 - 1 \right]$$

means that $k(z)$ is composed of the mappings

$$s = \frac{1 + z}{1 - z}; \qquad t = s^2; \qquad w = \frac{1}{4}(t - 1);$$

in that order. The first, a *linear fractional transformation*, maps D univalently onto the right half of the s-plane. The mapping $t = s^2$ is one-to-one *when restricted to the right half-plane*; its image is the entire t-plane minus the nonpositive real axis. The last mapping is simpler yet: a translation
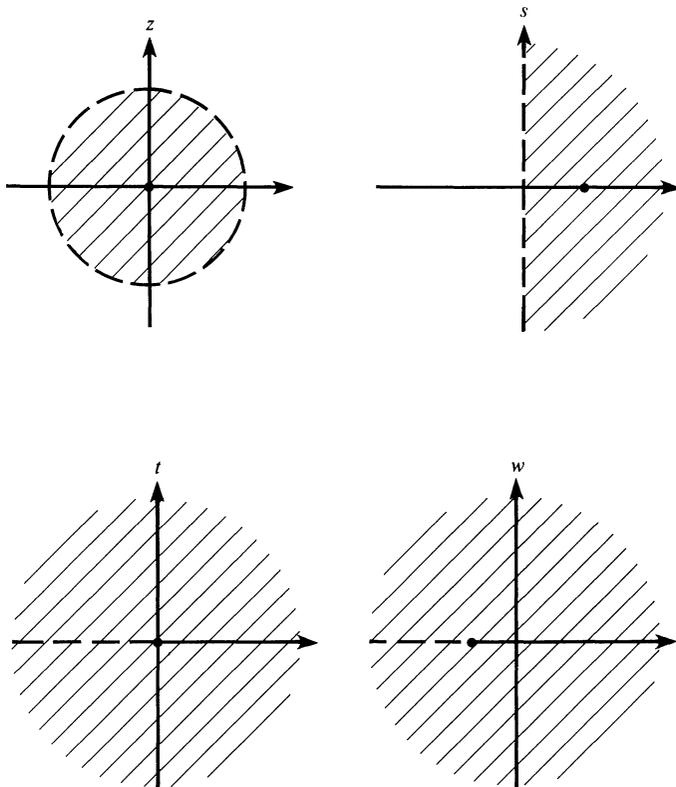
FIGURE 2

one unit to the left followed by a dilation with factor $1/4$. (See FIGURE 2.)

Could the Koebe function be unique among S-functions in having the largest possible coefficients? The answer is "essentially, but not quite": given one S-function, here is a way to construct infinitely many others with coefficients of the same modulus: if $f(z)$ is in S, and α is any real number, let

$$f_\alpha(z) = e^{-i\alpha}f(e^{i\alpha}z).$$

The f_α are called **rotations** of f, because the mapping $z \to e^{i\alpha}z$ is geometrically a counterclockwise rotation of $\mathbb{C}$, about $z = 0$, through α radians. Thus, f_α is formed by preceding and following f with opposite rotations through α radians. (The last operation keeps k_α in S.) In power series notation, if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$f_\alpha(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

where $b_n = a_n e^{i(n-1)\alpha}$. Since $|e^{i\alpha}| = 1$, $|a_n| = |b_n|$. The BC, which we can now state precisely, says that the extremal function k is unique "up to rotations":

THE BIEBERBACH CONJECTURE. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in S, then $|a_n| \leqslant n$ for every $n \geqslant 2$. If, for any n, $|a_n| = n$, then f is a rotation of the Koebe function.*

What does the BC mean? Why is it plausible?

Consider this infinite collection of optimization problems: for each $n \geqslant 2$, find in S a function $f(= z + \sum_{n=2}^{\infty} a_n z^n)$ for which $|a_n|$ is as large as possible. Two questions occur immediately:

1. Is there, for each n, an absolute bound on $|a_n|$, as f ranges over S?
2. If the nth coefficients of S-functions *are* bounded, is there an S-function f whose nth coefficient a_n *attains* that bound?

The answer to both questions is yes. Our first goal is to understand why (without, of course, assuming the BC).

Why are the nth coefficients of S-functions bounded? As a thought-experiment, consider what restriction, if any, the convergence of the power series $\sum a_n z^n$ of an S-function imposes on the a_n. The Cauchy-Hadamard formula for the radius of convergence of a power series implies that for every $R > 1$, $\lim_{n \to \infty} a_n / R^n = 0$. (Briefly, $|a_n| < R^n$ *asymptotically* as $n \to \infty$.) Although the Cauchy-Hadamard condition limits how fast the sequence $\{a_1, a_2, a_3, \ldots\}$ can grow, it is no restriction at all on any *particular* coefficient. Every polynomial, for example, is a convergent power series, and the coefficients of polynomials can be any complex numbers at all. Our experiment is over, but we learn that analyticity alone does not explain the boundedness of coefficients of S-functions. We need to look at univalence, a stronger and more subtle property in the complex case than in the real.

To illustrate the difference between the real and complex settings, consider whether there is an interesting "real-variable BC." Let $f(x) = x + \sum_{n=2}^{\infty} a_n x^n$ be a one-to-one *real-analytic* function, defined for all x in the real interval $(-1, 1)$; the a_n are *real* constants. Is there any restriction on the size of, say, a_3? The answer is no: if a_3 is any positive number, the polynomial $x + a_3 x^3$ is univalent. In fact, every odd polynomial with positive coefficients is monotone increasing, so no bounds apply to any of the odd-indexed coefficients. *Conclusion*: the BC is essentially a complex result—it has no interesting real-variable analogue.

What, then, is the connection between univalence (of a complex function) and the size of the a_n? A first observation is that the complex monomial z^n is an n-to-one function on D. (Notice how the real monomial x^n on $(-1, 1)$ differs!) Intuitively speaking, the multivalent powers z^n threaten to swamp the univalent term z if the a_n are too large. Here is a proposition that illustrates the general idea.

PROPOSITION. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$ *be a polynomial of degree n. If f is univalent in D, then $|a_n| \leqslant 1/n$.*

Proof. Consider $f'(z) = 1 + 2a_2 z + \cdots + na_n z^{n-1} = na_n(\frac{1}{na_n} + \cdots + z^{n-1})$. By the fundamental theorem of algebra, the polynomial within parentheses has $n - 1$ complex roots $c_1, \ldots, c_{n-1}$ (some may be repeated). Hence, $f'(z)$ can be factored:

$$f'(z) = na_n(z - c_1)(z - c_2) \cdots (z - c_{n-1}).$$

Since f is univalent on D, $f'(z)$ has no roots in D (compare the real case!) Therefore each c_i lies outside D; i.e., $|c_i| \geqslant 1$ for each i. Since f is in S,

$$1 = |f'(0)| = |na_n||c_1||c_2| \cdots |c_{n-1}| \geqslant |na_n|,$$

as claimed.

The converse is false, but:

Exercise. Let $f(z) = z + a_n z^n$. Show that $f(z)$ is univalent on D if and only if $|a_n| \leqslant 1/n$.

The key to questions 1 and 2 is the fact that S is a *compact* subset of the space of all analytic functions on D. The topology in question is that of **uniform convergence on compacta**: a sequence (f_n) of analytic functions on D converges to an analytic function f if (f_n) converges to f uniformly on every compact subset of D. The functional T that associates to an analytic function
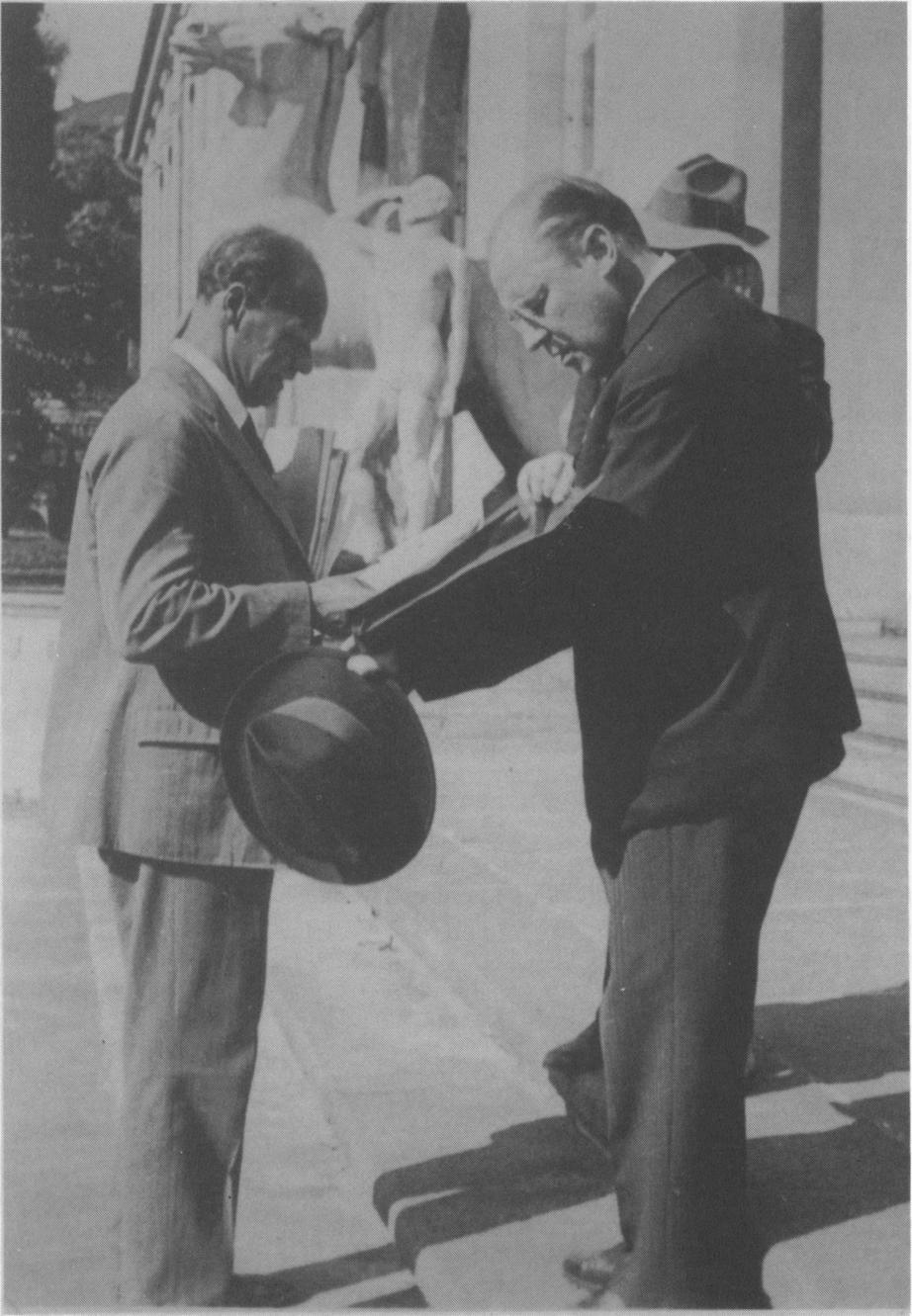
Bieberbach (on the left) with Sierpiński at the Zürich International Congress in 1932.

on D its nth Maclaurin coefficient:

$$T(f) = T\left(\sum a_n z^n\right) = a_n,$$

is continuous in this topology. Therefore, T *attains* a maximum modulus somewhere on S.

We will not attempt a rigorous discussion of compactness in spaces of analytic functions. Roughly speaking, S is compact because it is closed and **locally bounded** (or **normal**). "Closed" means that the limit of a convergent sequence of S-functions is again an S-function. The nontrivial part of this property of S is a standard theorem of A. Hurwitz: the limit of a convergent sequence of schlicht functions is either schlicht or constant. (The normalizations in S rule out the latter possibility.) "Locally bounded" means that for every r in $(0,1)$, there is a positive number $M(r)$ so that $|f(z)| < M(r)$ whenever $|z| < r$ and f is in S. Curiously, the local boundedness of S follows from Bieberbach's second coefficient theorem—a uniform bound on the second coefficient leads to bounds on all the others. This is shown in the next section.

Granted that bounds on the coefficients of S-functions are attained, why should the Koebe function attain them? Here are several ways in which k is the "largest" member of S.

Consider first the image domain $k(D)$ $(= \mathbb{C} - \{x : x \leqslant -1/4\})$. It is as "big" as it can be: adding any open set to $k(D)$ would introduce some overlap, thereby destroying the schlicht property of k. Another extremal property of $k(D)$ has to do with the distance $1/4$ between 0 and the boundary of $k(D)$ in the w-plane. In 1907, Koebe showed that the image $f(D)$ under any mapping in S contains a disk $\{|w| < \rho\}$ of some fixed radius ρ, independent of f. Bieberbach deduced from his second coefficient theorem that (as Koebe conjectured) $1/4$ is the largest possible value of ρ, attained only by k and its rotations. This result, known as **Koebe's one-quarter theorem**, says that the distortion of the domain D after mapping by an S-function is not too severe: the boundary of the image domain cannot approach the origin too closely. Since the boundary of $k(D)$ misses the origin by precisely $1/4$, the Koebe function exhibits the maximum legal distortion. So do rotations $k_\alpha(z) = e^{-i\alpha} k(e^{i\alpha} z)$ of the Koebe function; the image $k_\alpha(D)$ is simply $k(D)$ rotated $-\alpha$ radians about the origin. The symmetry of $k(D)$ is further circumstantial evidence for its extremality. Like D itself (which is the image of D under the *other* extremal S-mapping $f(z) = z$) $k(D)$ exhibits the simplicity and regularity characteristic of extremal objects.

Analytic as well as geometric intuition suggests that we should pick on the Koebe function. For example, the Maclaurin coefficients of a function f are proportional to f's derivatives at the origin. At one extreme—$f(z) = z$—there is no distortion; all higher derivatives vanish. Since for $k(z) = \sum nz^n$ the distortion (in the sense of the one-quarter theorem) is maximal, it is natural to guess that k represents the other extreme.

This section ends with a *caveat*: Informal evidence for the BC is easy to find. The evidence is valid, but the ease of finding it is misleading; rigorous proofs, as history shows, are much harder to find.

The second coefficient

The previous section says—informally—that for each $n \geqslant 2$, the problem of maximizing $|a_n|$ among S-functions has a solution, which is probably k. It is time to prove something. Bieberbach's second-coefficient theorem is the first concrete evidence for the general conjecture. Its corollaries (especially the *distortion theorem*, which implies that S is compact) are the basic tools for further study of the BC. The proof illustrates standard ideas and techniques of univalent function theory and shows how the Koebe functions arise as extrema.

BIEBERBACH'S THEOREM (1916). *If* $f = z + \sum_{n=2}^\infty a_n z^n$ *is in* S, *then* $|a_2| \leqslant 2$. *Equality holds if and only if* $f(z)$ *is a rotation of the Koebe function.*

A basic way to obtain Bieberbach-type inequalities is to relate power series coefficients to the (nonnegative) area of some region in the plane. The first such result is the **area theorem**, proved in

1914 by T. H. Gronwall. It refers not to S but to a related class of schlicht functions:

Definition. Let Σ denote the class of functions

$$g(z) = z + b_0 + b_1/z + \cdots = z + \sum_{n=0}^{\infty} b_n z^{-n}$$

that are analytic and univalent in $\Delta = \{z: |z| > 1\}$.

(Σ-functions are normalized to have a simple pole with residue one at infinity.) If g is in Σ, let $E = \mathbb{C} - g(\Delta)$ be the *complement* of the image domain. (See FIGURE 3.)
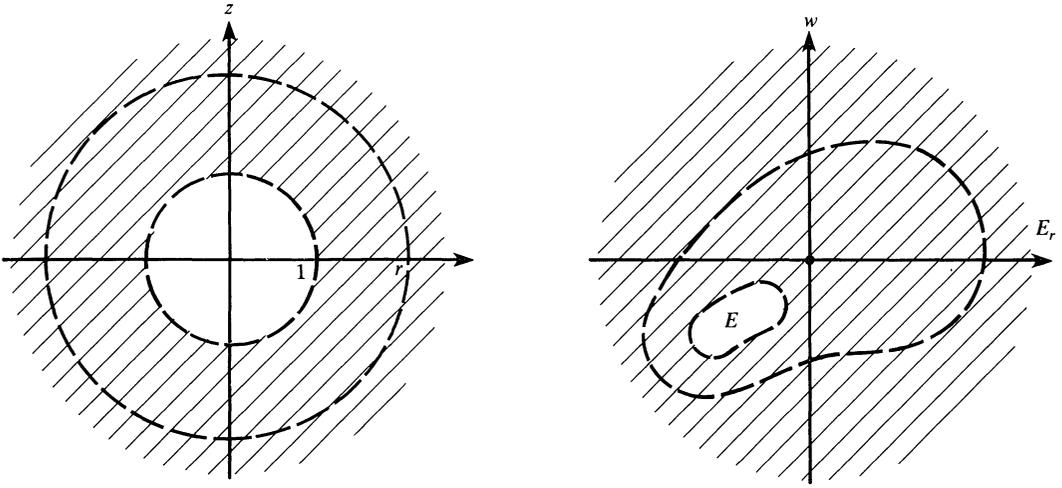
FIGURE 3

One hopes to calculate E's "area" in terms of the b_n. The quotes are necessary because E can be quite irregular. The solution is to approximate E from outside by nice domains $E(r) = \mathbb{C} - \{g(z): |z| > r\}$, and then *define*

$$\text{area } E = \lim_{r \to 1^+} \text{area } E(r).$$

Because the $E(r)$ have smooth (actually, analytic) boundary curves $\gamma(r)$, they have sensible areas which can be computed using Green's theorem in complex form. Explicitly, let $g(z) = w = u + iv$. Then

$$\frac{1}{\pi} \text{ area } E(r) = \frac{1}{\pi} \iint_{E(r)} du\, dv$$

$$= \frac{1}{2\pi i} \iint_{E(r)} d\overline{w}\, dw,$$

since $du\, dv = (1/2i)\, d\overline{w}\, dw$. By Green's theorem,

$$\frac{1}{\pi} \text{ area } E(r) = \frac{1}{2\pi i} \int_{\gamma(r)} \overline{w}\, dw$$

$$= \frac{1}{2\pi i} \int_{|z|=r} \overline{g}(z)\, g'(z)\, dz$$

$$= \frac{1}{2\pi} \int_0^{2\pi} re^{it} g'(re^{it}) \overline{g(re^{it})} \, dt,$$

where the changes of variable $w = g(z)$ and $z = re^{it}$ were made. (Note: the univalence of g was used here!) Writing g and g' as power series,

$$\frac{1}{\pi} \text{ area } E(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(re^{it} + \sum_{n=1}^{\infty} nb_n r^{-n} e^{-int} \right) \left(re^{-it} + \sum_{n=0}^{\infty} \bar{b}_n r^{-n} e^{int} \right) dt$$

$$= r^2 - \sum_{n=1}^{\infty} r^{-2n} n |b_n|^2,$$

by the orthogonality of distinct powers of e^{it}. Since area $E(r) \geqslant 0$, the partial sum

$$\sum_{n=1}^{m} r^{-2n} n |b_n|^2 \leqslant r^2$$

for every $m > 0$. Letting $r \to 1^+$, we have

$$\sum_{n=1}^{m} n |b_n|^2 \leqslant 1, \qquad m = 1, 2, \ldots.$$

We have proved:

AREA THEOREM. *If $g = z + \sum_{n=0}^{\infty} b_n z^{-n}$ is in Σ, then $\sum_{n=1}^{\infty} n|b_n|^2 \leqslant 1$.*

COROLLARY. *If $g(z)$ is in Σ, then $|b_1| \leqslant 1$. Equality holds if and only if $g(z) = z + b_0 + e^{i\alpha}/z$ (where $|e^{i\alpha}| = 1$).*

The area theorem is about Σ-functions but it leads indirectly to coefficient estimates for the class S. Beginning with an S-function, one applies algebraic transformations to concoct a Σ-function, keeping track of the coefficients. This is the idea of Bieberbach's proof.

Proof of Bieberbach's theorem. Given $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in S, construct the auxiliary functions

$$g(z) = \left(f(z^2) \right)^{1/2} \quad \text{and} \quad h(z) = 1/g(1/z).$$

(The function $f(z^2)$ has a schlicht square root because f is univalent.) A routine calculation shows that $h(z)$ is in Σ, and has Laurent series

$$h(z) = z - \frac{a_2}{2z} + \cdots.$$

The corollary to the area theorem implies that $|a_2| \leqslant 2$; equality holds if and only if $h(z) = z + b/z$, where $|b| = 1$. Unravelling the definition of h in terms of f shows that

$$h(z) = z + \frac{b}{z} \quad \text{if and only if} \quad f(z) = \frac{z}{(1 + bz)^2}.$$

The function $f(z)$ is, as claimed, a rotation of the Koebe function.

The principle observed above—to start with an S- or Σ-function, carry out some algebraic transformation, and then apply a known coefficient theorem to the result—yields several interesting consequences of Bieberbach's theorem. Some of them give as much insight into the BC as the second coefficient theorem itself. A sampler of such results follows.

KOEBE'S ONE-QUARTER THEOREM. *Let $f = z + a_2 z^2 + \cdots$ be in S, and suppose that $f(D)$ omits the value w_0 (i.e., $f(z) \neq w_0$ if $|z| < 1$). Then $|w_0| \geqslant 1/4$; equality can hold if and only if f is a rotation of the Koebe function.*

Proof. The transformed function

$$g(z) = \frac{w_0 f(z)}{w_0 - f(z)} = h(f(z)), \quad \text{where } h(w) = \frac{w_0 w}{w_0 - w}$$

is univalent because h is (on $\mathbb{C} - \{w_0\}$). Computing g's derivatives at the origin yields

$$g(z) = z + \left(a_2 + \frac{1}{w_0}\right)z^2 + \cdots .$$

Thus, g is in S and by Bieberbach's theorem, $|a_2 + 1/w_0| \leq 2$. By the triangle inequality, $|w_0| \geq 1/4$; equality is possible only if $|a_2| = 2$. In that case, f is a rotation of k.

Bieberbach's theorem says that for an S-function f, $|f''(0)| \leq 4$. This information can be transferred from the origin to any z_0 in D by composition with the linear fractional transformation

$$A(w) = \frac{w + z_0}{1 + w z_0}.$$

Because A is a schlicht mapping of D onto itself, with $A(0) = z_0$, the composite $f(A(w))$ is also schlicht, though not normalized. Setting

$$h(w) = \frac{f(A(w)) - f(z_0)}{(1 - |z_0|^2) f'(z_0)}$$

accomplishes the normalization; a messy (but explicit) calculation shows that

$$h(w) = w + \left[\frac{1}{2}(1 - |z_0|^2)\frac{f''(z_0)}{f'(z_0)} - \bar{z}_0\right]w^2 + \cdots .$$

From Bieberbach's theorem (dropping the subscript on z),

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{4|z|}{1 - |z|^2}, \qquad |z| < 1.$$

The last inequality can be "integrated" to give upper and lower bounds on $|f'(z)|$ and $|f(z)|$ in terms of $|z|$:

DISTORTION THEOREM. *If f is in S and $|z| < 1$, then*

(1)
$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} ;$$

(2)
$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2} .$$

("Distortion" in this context refers to the fact that the mapping f magnifies—distorts—small distances near z by the factor $|f'(z)|$.)

Proof. With $z = re^{i\alpha}$, the inequality before the theorem becomes

$$\frac{4}{1 - r^2} \geq \left| e^{i\alpha}\frac{f''(re^{i\alpha})}{f'(re^{i\alpha})} - \frac{2r}{1 - r^2} \right| = \left| \frac{\partial}{\partial r}\log[(1 - r^2)f'(re^{i\alpha})] \right|;$$

the last equality is checked by direct calculation. Now we integrate along the ray from 0 to z; since $f'(0) = 1$,

$$\left| \log[(1 - r^2)f'(re^{i\alpha})] \right| = \left| \int_0^r \frac{\partial}{\partial \rho}\left[\log[(1 - \rho^2)f'(\rho e^{i\alpha})]\right] d\rho \right|$$

$$\leq \int_0^r \left| \frac{\partial}{\partial \rho}\log[(1 - \rho^2)f'(\rho e^{i\alpha})] \right| d\rho$$

$$\leqslant \int_0^r \frac{4}{1-\rho^2}\,d\rho = 2\log\frac{1+r}{1-r}.$$

Taking the real part of $\log[(1-r^2)f'(re^{i\alpha})]$ gives

$$-2\log\frac{1+|z|}{1-|z|} \leqslant \log\big[(1-|z|^2)|f'(z)|\big] \leqslant 2\log\frac{1+|z|}{1-|z|},$$

which is equivalent to (1). The proof of (2) is similar.

The distortion theorems are basic to all further analysis of schlicht functions. The Koebe function plays its usual role: all of the inequalities are strict unless f is a rotation of k.

Now we can settle the *existence* question raised above of bounds for individual coefficients of S-functions. What remains undone from the previous section is to show that S is locally bounded. This is exactly what the right side of (2) means:

$$\text{if } |z| < r, \text{ then } |f(z)| \leqslant k(r) = \frac{r}{(1-r)^2}.$$

At last we know that coefficient bounds exist. One can also derive (admittedly unprepossessing) coefficient *estimates* from (2):

Exercise. Show that $|a_n| \leqslant n^2 e^2/4$, $n = 2, 3, \ldots$, for every f in S.

Hint. Use **Cauchy's inequalities**: if $f(z) = \Sigma a_n z^n$ is analytic and $|f(z)| \leqslant M$ for $|z| < r$, then $|a_n| \leqslant M/r^n$.

We will do much better for all coefficients in the next section. As a last corollary of Bieberbach's theorem, though, let's improve the a_3 estimate (the exercise gives $|a_3| \leqslant 16.63$).

PROPOSITION. *If f is in S, then $|a_3| \leqslant 5$.*

Proof. The function $g(z) = 1/f(1/z) = z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots$ is in Σ. By the area theorem, $|a_2^2 - a_3| \leqslant 1$, so

$$|a_3| \leqslant |a_2^2 - a_3| + |a_2|^2 \leqslant 1 + 4 = 5.$$

Some history, two classical proofs, and de Branges' solution

The Riemann mapping theorem guarantees the *existence* of a schlicht mapping from D onto any given simply connected proper subdomain of the complex plane. Univalent function theory is the study of *concrete* properties (especially extremal properties) of "Riemann maps." The primeval fact is that S is *normal*; as discussed above, this assures that extremal problems (e.g., to maximize the nth coefficient) have solutions. This fundamental theorem was first proved in 1907 by Koebe, arguably the father of the discipline.

That the Koebe function plays some special role in S was evident before the BC appeared in 1916. For example, in 1914 Gronwall proved (among other properties of k) that if f is in S and $|z| < 1$, then

$$|k(-|z|)| \leqslant |f(z)| \leqslant |k(|z|)|,$$

which means that on every circle $|z| = r$, $|k(z)|$ attains both the largest maximum and the smallest minimum of any S-function. (This is also a corollary of the distortion theorem.) With so much evidence for the extremality of k in S, the conjecture that k also maximizes individual coefficients was certainly in the wind. Bieberbach's contribution is more in having proved the second coefficient theorem and its corollaries than in having issued the conjecture itself. (See also [11], ch. 2.)

Progress on the BC occurred in several directions. Here are three "genres" of partial results:
1. $|a_n| \leqslant n$ for specific n;
2. $|a_n| \leqslant n$ for subclasses of S;
3. $|a_n| \leqslant Cn$ for sufficiently large C.

(See [3] and [7] for progress in other directions.)

Results of the first type came slowly. The third coefficient theorem—$|a_3| \leqslant 3$—was proved in 1923 by the Czech-educated mathematician K. Loewner, who later emigrated to the United States. The proof is deep, delicate, and completely different from Bieberbach's second-coefficient proof. Loewner's partial differential equation method is notable both for having been found so early and because it figures in de Branges' proof of the general conjecture. No more "$|a_n| \leqslant n$" theorems were proved for more than 30 years, and then by still different methods. In 1955, P. Garabedian and M. Schiffer, who were then Loewner's colleagues at Stanford, used a specially developed calculus of variations in S to prove that $|a_4| \leqslant 4$. The sixth- and fifth-coefficient theorems followed in 1968 and 1972. By 1984, earlier complicated proofs had been revised and shortened, but the conjecture remained open for all n greater than six.

Two early "subclass" theorems, due to R. Nevanlinna (1920) and to W. Rogosinski and J. Dieudonné (independently, around 1930), respectively, assert that the BC holds for S-functions with either (i) starlike range ($f(D)$ contains the segment joining any of its points to the origin) or (ii) all real coefficients. (Loewner had proved a special case of (i) in 1917.) In a slightly different direction, certain subclasses of S were shown to satisfy more stringent coefficient growth estimates. For example, J. Clunie and Ch. Pommerenke showed in 1966 that the coefficients of *bounded* S-functions satisfy the order-of-growth estimate

$$|a_n| = O(n^{-1/2 - \alpha})$$

for some positive number α (of unknown best value). Other recent work showed that $|a_n| \leqslant n$ for S-functions for which (iii) $|a_2| \leqslant 1.05$; (iv) f is "near" the Koebe function in an appropriate topology; or (v) f is "far" from k. As an example, we prove (ii).

PROPOSITION. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in S, with each a_n real. Then $|a_n| \leqslant n$.*

Proof. Because the a_n are real, the identity

$$f(\bar{z}) = \sum a_n \bar{z}^n = \sum \overline{a_n z^n} = \overline{f(z)}$$

holds for all z in D. It follows that the image domain $f(D)$ is symmetric about the real axis; in particular, $f(z)$ is real if and (because f is univalent) only if z is real. Moreover, since $f'(0) = 1$, f maps the upper (resp. lower) half of D to the upper (resp. lower) half of $f(D)$. (See FIGURE 4.)
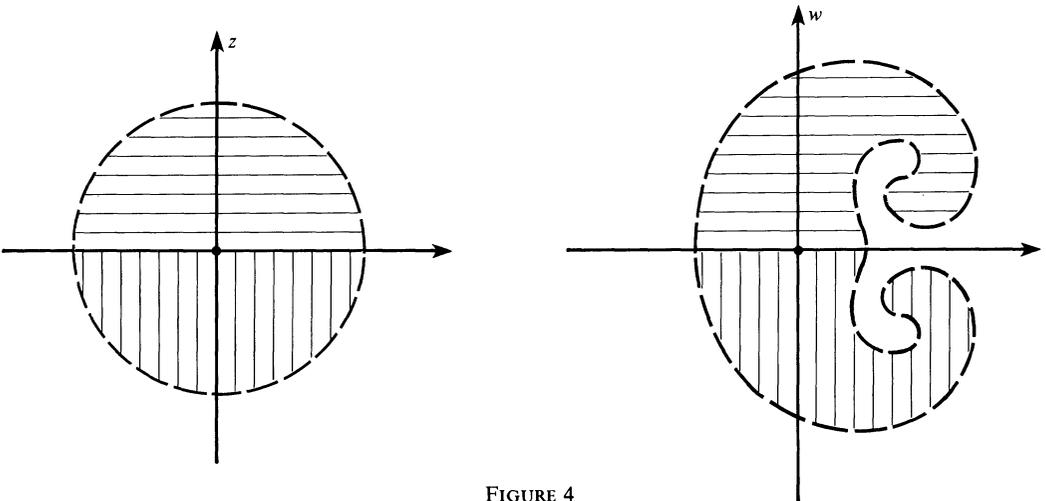

FIGURE 4

Writing f as a power series in polar coordinates (with $a_1 = 1$), we have

$$f(z) = f(re^{i\alpha}) = \sum_{k=1}^{\infty} a_k r^k e^{ik\alpha}$$

$$= \sum a_k r^k \cos(k\alpha) + i \sum a_k r^k \sin(k\alpha)$$

$$= U(z) + iV(z),$$

where U and V are *real-valued* functions of z. Recall from elementary calculus that if n and m are integers, functions of either form $\sin(n\alpha)$ or $\cos(m\alpha)$ are pairwise orthogonal with respect to integration in α over the interval $[-\pi, \pi]$, but that

$$\int_{-\pi}^{\pi} [\sin(n\alpha)]^2 \, d\alpha = \pi.$$

Hence, for each $r < 1$,

$$|a_n r^n| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} \sin(n\alpha) V(re^{i\alpha}) \, d\alpha \right|.$$

Because the integrand is an *even* function of α,

$$|a_n r^n| = \frac{2}{\pi} \left| \int_{0}^{\pi} \sin(n\alpha) V(re^{i\alpha}) \, d\alpha \right|$$

$$\leqslant \frac{2}{\pi} \int_{0}^{\pi} |\sin(n\alpha) V(re^{i\alpha})| \, d\alpha$$

$$= \frac{2}{\pi} \int_{0}^{\pi} |\sin(n\alpha)| V(re^{i\alpha}) \, d\alpha;$$

the last equality follows from the mapping property of f — $V(z)$ is nonnegative if α lies in $[0, \pi]$. Now we invoke a possibly unfamiliar but straightforward inequality of real analysis (see, e.g., [**12**, p. 356] for a proof):

$$|\sin(n\alpha)| \leqslant n \sin(\alpha), \qquad 0 \leqslant \alpha \leqslant \pi, \qquad n = 1, 2, \ldots.$$

Given this,

$$|a_n r^n| \leqslant \frac{2}{\pi} n \int_{0}^{\pi} \sin(\alpha) V(re^{i\alpha}) \, d\alpha$$

$$= \frac{1}{\pi} n \int_{0}^{2\pi} \sin(\alpha) V(re^{i\alpha}) \, d\alpha$$

$$= na_1 = n.$$

Because $r < 1$ is arbitrary, we can let r tend to 1 from below to complete the proof.

The order of growth of the a_n was known early to be linear, as conjectured. The first good theorem of type $|a_n| \leqslant Cn$ appeared in 1925, when J. E. Littlewood proved that for f in S,

$$|a_n| \leqslant en.$$

The main ingredient is Littlewood's integral estimate (see [**8**, p. 38]): if $0 < r < 1$ and f is in S, then

$$\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\alpha})| \, d\alpha \leqslant \frac{r}{1-r}.$$

Assuming this, the proof is straightforward. Let $f(z) = \sum a_n z^n$; then

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz.$$

M. M. Schiffer, S. Bergman, M. Protter, J. Herriot and C. Loewner at Stanford in the 1950's. Schiffer and Loewner played important roles in the history of the Bieberbach problem.

(This simplest version of the Cauchy integral theorem can be checked by direct computation.) Writing $z = re^{i\alpha}$ and $dz = ire^{i\alpha} \, d\alpha$,

$$|a_n| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(re^{i\alpha})}{r^n e^{in\alpha}} \, d\alpha \right|$$

$$\leqslant \frac{1}{2\pi r^n} \int_0^{2\pi} |f(re^{i\alpha})| \, d\alpha \leqslant \frac{1}{r^{n-1} - r^n}.$$

This inequality holds for *every* r in $(0,1)$. Elementary calculus shows that the right side attains its minimum when $r = 1 - 1/n$. Substituting this value of r into the last inequality yields

$$|a_n| \leqslant \frac{1}{r^{n-1} - r^n} = n \left(1 + \frac{1}{n-1} \right)^{n-1} < en.$$

In 1974, A. Baernstein improved Littlewood's integral inequality by showing that k has the largest possible integral mean of every order: for each f in S and each real number p,

$$\int_0^{2\pi} |f(re^{i\alpha})|^p \, d\alpha \leqslant \int_0^{2\pi} |k(re^{i\alpha})|^p \, d\alpha.$$

With $p = 1$, calculating the right side gives

$$\int_0^{2\pi} |f(re^{i\alpha})| \, d\alpha \leqslant \frac{r}{1 - r^2}.$$

Exercise. Mimic the previous proof to show that

$$|a_n| \leqslant \frac{e}{2} n \approx 1.36 \, n.$$

Given any S-function f, setting $g(z) = f(z^2)^{1/2} = z + \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$ produces an *odd* univalent function. (We did this already in the proof of Bieberbach's theorem.) If $f = k$, then

$$k(z^2)^{1/2} = z + z^3 + z^5 + z^7 + \cdots,$$

i.e., $b_k = 1$ for all k. In 1932, J. Littlewood and R. Paley made the natural conjecture that $|b_k| \leqslant 1$ for every odd S-function. (They *proved* that $|b_k| < 14$.)

Exercise. By inverting the square-root transform, show that the Littlewood-Paley conjecture implies the BC.

The Littlewood-Paley conjecture was disproved the following year by M. Fekete and G. Szegő, but it led M. Robertson to the slightly weaker conjecture that $|b_k| \leqslant 1$ in an average sense:

ROBERTSON CONJECTURE (1936). *If* $g(z) = z + b_3 z^3 + b_5 z^5 + \cdots$ *is in* S, *then* $1 + |b_3|^2 + |b_5|^2 + \cdots + |b_{2n-1}|^2 \leqslant n.$

The Robertson conjecture still implies the BC, even in a strong form (see, e.g., [**8**, p. 66]). It is what de Branges indirectly proved.

Iterating the square-root transform leads to looking at roots of higher order, and in a limiting sense, to the logarithm of a schlicht function. In 1939, H. Grunsky obtained a new class of coefficient inequalities that would prove, much later, to be an important step toward a general solution of the BC. Beginning with an S-function f, Grunsky studied the coefficients c_{jk} of the function

$$\log \frac{f(z) - f(w)}{z - w} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} z^j w^k,$$

analytic in the *two* complex variables z and w. The **Grunsky inequalities** are an infinite family of estimates on the "logarithmic coefficients" c_{jk}. (Formally, they assert that the infinite matrices

(c_{jk}) associated to S-functions are bounded by one in a suitable norm.)

Given the Grunsky inequalities, a difficult question remains: how to use knowledge of the logarithmic coefficients c_{jk} to estimate the ordinary Taylor coefficients a_n of f, i.e., how to "exponentiate" the Grunsky inequalities. Different techniques were developed in the sixties and seventies by N. A. Lebedev and I. M. Milin in the USSR and by C. FitzGerald in the US. They were applied to improve earlier "$|a_n| \leqslant Cn$" theorems; by 1978, D. Horowitz had shown $|a_n| < 1.0657n$. (See [13] for a chronology of efforts to reduce C.)

Lebedev and Milin's work led to a conjecture of exceptional interest:

LEBEDEV-MILIN CONJECTURE (1967). *If f is in S and $\log f(z)/z = \sum_{k=1}^{\infty} c_k z^k$, then*

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|c_k|^2 - \frac{4}{k} \right) \leqslant 0, \qquad n = 1, 2, 3, \dots.$$

Since for the Koebe function, $\log k(z)/z = \sum_{k=1}^{\infty} (2/k) z^k$, the obvious guess is that for every f in S, $|c_k| \leqslant 2/k$ for every k. This is too optimistic—it can be shown to imply the Littlewood-Paley conjecture. The Lebedev-Milin conjecture, like the Robertson conjecture, says that, nevertheless, the obvious guess is true in a weaker, average sense for the first n coefficients. Lebedev and Milin showed that their conjecture implies the Robertson conjecture, which is stronger than the BC. Thus some experts were surprised that the Lebedev-Milin conjecture is exactly what de Branges proved.

De Branges' proof has two main ingredients—a system of time-dependent weight functions, and the Loewner differential equation (part of the classical third coefficient proof), which also involves a time parameter. De Branges combines these to construct a time-dependent form of the Lebedev-Milin inequality which is clearly valid at time $t = \infty$ and which *equals* the Lebedev-Milin inequality at $t = 0$. The brilliant (and "miraculous," according to one expert) choice of the weighting functions assures that the inequality remains valid as t retreats to zero. The necessary property of the weight functions reduces to an inequality on generalized hypergeometric polynomials.

The computer played a role in de Branges' search for a solution, though not in his ultimate proof. At a late stage in his work, de Branges approached W. Gautschi of the computer science department at Purdue for help in testing his conjectured inequality numerically. The spectacular results—proofs of the BC for the first thirty coefficients—were encouraging evidence that de Branges' general method would work. They also led Gautschi to call R. Askey, at the University of Wisconsin, to inquire whether the inequality had been proven earlier. Surprisingly, it had, in a 1976 paper of Askey and G. Gasper [1]. By remarkable coincidence, a property of special functions of a *real* variable finished the proof of a *complex* result.

De Branges' originally lengthy proof was at first received skeptically. Incorrect proofs of the BC had been announced before. Nevertheless, the new proof was confirmed by the Leningrad Seminar in Geometric Function Theory (of which I. M. Milin is a member), in five marathon sessions in April and May, 1984. A preliminary version of the proof was issued in preprint form [6]. The proof was later revised and shortened; now, some of the individual coefficient proofs are longer and more difficult. De Branges' version was published in 1985 [5]. (See also [15], [10], and [9] for a slightly different version of the proof and for more detailed commentary.)

Mathematicians working on the BC created and advanced theories (including Schiffer's variational method and the theory of quadratic differentials) which, though not necessarily used in de Branges' solution, have found applications elsewhere. This is not to say that de Branges does not use earlier work. Loewner's method, the Lebedev-Milin conjecture (and, implicitly, the Grunsky inequalities), and Askey and Gasper's theorem are all essential—along with de Branges' unique contribution—to the ultimate solution.

Though its most famous problem has been solved, important questions in geometric function theory remain. For example, sharp coefficient estimates are unknown for normalized p-to-one analytic functions, for Σ-functions, and for S-functions with growth restrictions. De Branges'

contribution was celebrated at an international symposium held at Purdue in March, 1985; many new problems and directions for research were proposed. At the University of Maryland, 1985–1986 was declared a special year in complex analysis. How and where de Branges' special techniques may apply remains to be seen, but his achievement seems certain to spur new work and interest in special functions, optimization theory, functional analysis, and complex analysis.

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