HISTORY OF HOMOLOGICAL ALGEBRA

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Homological algebra had its origins in the 19th century, via the work of Riemann (1857) and Betti (1871) on "homology numbers," and the rigorous development of the notion of homology numbers by Poincaré in 1895. A 1925 observation of Emmy Noether [N25] shifted the attention to the "homology groups" of a space, and algebraic techniques were developed for computational purposes in the 1930's. Yet homology remained a part of the realm of topology until about 1945.

During the period 1940-1955, these topologically-motivated techniques for computing homology were applied to define and explore the homology and cohomology of several algebraic systems: Tor and Ext for abelian groups, homology and cohomology of groups and Lie algebras, and the cohomology of associative algebras. In addition, Leray introduced sheaves, sheaf cohomology and spectral sequences.

At this point Cartan and Eilenberg's book [CE] crystallized and redirected the field completely. Their systematic use of derived functors, defined via projective and injective resolutions of modules, united all the previously disparate homology theories. It was a true revolution in mathematics, and as such it was also a new beginning. The search for a general setting for derived functors led to the notion of abelian categories, and the search for nontrivial examples of projective modules led to the rise of algebraic K-theory. Homological algebra was here to stay.

Several new fields of study grew out of the Cartan-Eilenberg Revolution. The importance of regular local rings in algebra grew out of results obtained by homological methods in the late 1950's. The study of injective resolutions led to Grothendieck's theory of sheaf cohomology, the discovery of Gorenstein rings and Local Duality in both ring theory and algebraic geometry. In turn, cohomological methods played a key role in Grothendieck's rewriting of the foundations of algebraic geometry, including the development of derived categories. Number theory was infused with new results from Galois cohomology, which in turn led to the development of étale cohomology and the eventual solution of the Weil Conjectures by Deligne.

Simplicial methods were introduced in the 1950's by Dold, Kan, Moore and Puppe. They led to the rise of homotopical algebra and nonabelian derived functors in the 1960's. Among its many applications, perhaps André-Quillen homology for commutative rings and higher algebraic K-theory are the most noteworthy. Simplicial methods also played a more recent role in the development of Hochschild homology, topological Hochschild homology and cyclic homology.

This completes a quick overview of the history we shall discuss in this article. Now let us turn to the beginnings of the subject.

Typeset by $\mathcal{AMS}$-TEX

Betti numbers, Torsion Coefficients and the rise of Homology

Homological algebra in the 19th century largely consisted of a gradual effort to define the "Betti numbers" of a (piecewise linear) manifold. Beginning with Riemann's notion of genus, we see the gradual development of numerical invariants by Riemann, Betti and Poincaré: the Betti numbers and Torsion coefficients of a topological space. Indeed, the subject did not really move beyond these numerical invariants until about 1930. And it was not concerned with anything except invariants of topological spaces until about 1945.

Riemann and Betti.

The first step was taken by Riemann (1826–1866) in his great 1857 work "Theorie der Abel'schen Funktionen" [Riem, VI]. Let C be a system of one or more closed curves C_j on a surface S, and consider the contour integral $\int_C X\,dx + Y\,dy$ of an exact differential form. Riemann remarked that this integral vanished if C formed the complete boundary of a region in S (Stokes' Theorem), and this led him to a discussion of "connectedness numbers." Riemann defined S to be $(n+1)$-*fold connected* if there exists a family C of n closed curves C_j on S such that no subset of C forms the complete boundary of a part of S, and C is maximal with this property. For example, S is "simply connected" (in the modern sense) if it is 1-fold connected. As we shall see, the connectness number of S is the homology invariant $1 + \dim H_1(S;\mathbb{Z}/2)$.

Riemann showed that the connectedness number of S was independent of the choice of maximal family C. The key to his assertion is the following result, which is often called "Riemann's Lemma" [Riem, p. 85]: Suppose that A, B and C are three families of curves on S such that A and B form the complete boundary of one region of S, and A and C form the complete boundary of a second region of S. Then B and C together must also form the boundary of a third region, obtained as the symmetric difference of the other two regions (obtained by adding the regions together, and then subtracting any part where they overlap).

If we write $C \sim 0$ to indicate that C is a boundary of a region then Riemann's Lemma says that if $A + B \sim 0$ and $A + C \sim 0$ then $B + C \sim 0$. This, in modern terms, is the definition of addition in mod 2 homology! Indeed, the C_j in a maximal system form a basis of the singular homology group $H_1(S;\mathbb{Z}/2)$.

Riemann was somewhat vague about what he meant by "closed curve" and "surface," but we must remember that this paper was written before Möbius discovered the "Möbius surface" (1858) or Peano studied pathological curves (1890). There is another ambiguity in this Lemma, pointed out by Tonelli in 1875: every curve C_j must be used exactly once.

Riemann also considered the effect of making cuts (*Querschnitte*) in S. By making each cut q_j transverse to a curve C_j (see [Riem, p. 89]), he showed that the number of cuts needed to make S simply connected equals the connectivity number. For a compact Riemann surface, he shows [Riem, p. 97] that one needs an even number $2p$ of cuts. In modern language, p is the *genus* of S, and the interaction between the curves C_j and cuts q_j forms the germ of Poincaré Duality for $H_1(S;\mathbb{Z}/2)$.

Riemann had poor health, and frequently visited Italy for convalescence between 1858 and his death in 1866. He frequently visited Enrico Betti (1823–1892) in

Pisa, and the two of them apparently discussed the idea of extending Riemann's construction to higher dimensional manifolds. Two documents with very similar definitions survive.

One is an undated "Fragment on Analysis Situs" [Riem, XXVIII], discovered among Riemann's effects, in which Riemann defines the n-dimensional connectedness of a manifold M: replace "closed curve" with n-dimensional subcomplex (*Streck*) without boundary, and "bounding a region" with "bounding an $(n+1)$-dimensional subcomplex). Riemann also defined higher dimensional cuts (submanifolds whose boundary lies on the boundary of M) and observed that a cut of dimension $\dim(M) - n$ either drops the n-dimensional connectivity by one, or raises the $(n-1)$-dimensional connectivity by one. In fairness, we should point out that Riemann's notion of connectedness is not independent of the choice of basis, because his notion that A and B are similar (*veränderlich*) is not the same as A and B being homologous; a counterexample was discovered by Heegaard in 1898.

The other document is Betti's 1871 paper [Betti]. The ideas underlying this paper are the same as those in Riemann's fragment, and Betti states that his proof of the independence of the homology numbers from the choice of basis is based upon the proof in Riemann's 1857 paper. However, Heegaard observed in 1898 that Betti's proof of independence is not correct in several respects, starting from the fact that a meridian on a torus is not closed in Betti's sense.

Betti also made the following assertion ([Betti, p.148]), which presages the Poincaré Duality Theorem: "In order to render a finite n-dimensional space simply connected, by removing simply connected sections, it is necessary and sufficient to make p_{n-1} linear cuts, ... , p_1 cuts of dimension $n-1$," where $p_i + 1$ is the ith connectivity number. Heegaard found mistakes in Betti's proof here too, and Poincaré observed in 1899 [Poin, p. 289] that the problem was in (Riemann and) Betti's definition of similarity: it is not enough to just consider the set underlying A, one must also account for multiplicities.

Poincaré and Analysis Situs.

Inspired by Betti's paper, Poincaré (1854–1912) developed a more correct homology theory in his landmark 1895 paper "Analysis Situs" [Poin]. After defining the notion, he fixes a piecewise linear manifold (*variété*) V. Then he considers formal integer combinations of oriented n-dimensional submanifolds V_i of V, and introduces a relation called a *homology*, which can be added like ordinary equations: $\sum k_i V_i \sim 0$ if there is an $(n+1)$-dimensional submanifold W whose boundary consists of k_1 submanifolds like V_1, k_2 submanifolds like V_2, etc.

Poincaré calls a family of n-dimensional submanifolds V_i linearly independent if there is no homology (with integer coefficients) connecting them. In honor of Enrico Betti, Poincaré defined the *nth Betti number* of V to be $b_n + 1$, where b_n is the size of a maximal independent family. Today we call b_n the nth Betti number, because it is the dimension of the rational vector space $H_n(V;\mathbb{Q})$. For geometric reasons, he did not bother to define the nth Betti number for $n = 0$ or $n = \dim(V)$.

With this definition, Poincaré stated his famous Duality Theorem [Poin, p. 228]: for a closed oriented (m-dimensional) manifold, the Betti numbers equally distant from the extremes are equal, viz., $b_i = b_{m-i}$. Unfortunately, there was a gap in Poincaré's argument, found by Heegaard in 1898. Poincaré published a new proof in 1899, using a triangulation of V and restricting his formal sums $\sum k_i V_i$ to linear combinations of the simplices in the triangulation. Of course this restriction yields "reduced" Betti numbers which could potentially be different from the Betti numbers he had defined in 1895. Using simplicial subdivisions, he sketched a proof that these two kinds of Betti numbers agreed. (His sketch had a geometric gap, which was filled in by J. W. Alexander in 1915.) This 1899 paper was the origin of the simplicial homology of a triangulated manifold.

Poincaré's 1899 paper also contains the first appearance of what would eventually (after 1929) be called a chain complex. Let V be an oriented polyhedron. On p. 295 of [Poin], he defined boundary matrices ε^q as follows. The (i,j) entry describes whether or not the jth $(q-1)$-dimensional simplex in V lies on the boundary of the ith q-dimensional simplex: $\varepsilon^q_{ij} = \pm 1$ if it is ($+1$ if the orientation is the same, -1 if not) and $\varepsilon^q_{ij} = 0$ if they don't meet. Poincaré called the collection of these matrices the *scheme* of the polyhedron, and demonstrated on p. 296 that $\varepsilon^{q-1} \circ \varepsilon^q = 0$. This is of course the familiar condition that the matrices ε^q form the maps in a chain complex, and today Poincaré's scheme is called the simplicial chain complex of the oriented polyhedron V.

Another major result in *Analysis Situs* is the generalization of the notion of Euler characteristic to higher dimensional polyhedra V. If α_n is the number of n-dimensional cells, Poincaré showed that the alternating sum $\chi(V) = \sum (-1)^n \alpha_n$ is independent of the choice of triangulation of V (modulo the gap filled by Alexander). On p. 288 he showed that $\chi(V)$ is the alternating sum of the Betti numbers b_n (in the modern sense); because of this result $\chi(V)$ is today called the *Euler-Poincaré characteristic* of V. Finally, when V is closed and $\dim(V)$ is odd, he used Duality to deduce that $\chi(V) = 0$.

In 1900, Poincaré returned once again to the subject of homology, in the *Second complément à l'Analysis Situs.* This paper is important from our perspective because it introduced linear algebra and the notion of torsion coefficients. To do this, Poincaré considered the sequence of integer matrices (or *tableaux*) T_p which describe the boundaries of the p-simplices in a polyhedron; this sequential display of integer matrices was the second occurrence of the notion of chain complex.

In Poincaré's framework, one performs elementary row and column operations upon all the matrices until the matrix T_p had been reduced to the block form

$$T_p = \begin{pmatrix} I & 0 & 0 \\ 0 & K_p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K_p = \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & \ddots \end{pmatrix}, \qquad 1 < k_1, k_1 | k_2, k_2 | k_3, \cdots$$

Here I denotes an identity matrix. The pth Betti number b_p is the difference between the number of zero columns in T_p and the number of nonzero rows in T_{p+1} [Poin, p. 349]. The pth *torsion coefficients* were defined as the integers k_1, k_2, etc. in the matrix K_{p+1} [Poin, p. 363].

In modern language, $H_n(V;\mathbb{Z})$ is a finitely generated abelian group, so it has the form $\mathbb{Z}^{b_n} \oplus \mathbb{Z}/k_1 \oplus \mathbb{Z}/k_2 \oplus \cdots$ with $k_1|k_2$, $k_2|k_3$, etc. Here b_n is the Betti number, and the pth torsion coefficients are the orders of the finite cyclic groups $\mathbb{Z}/k_i$. Of course, since homology was not thought of as a group until 1925 (see [N25]), this formulation would have looked quite strange to Poincaré!

Homology of topological spaces (1900–1935).

The next 25 years were a period of consolidation and clarification of Poincaré's ideas. For example, the Duality Theorem for the mod 2 Betti numbers, even for nonoriented manifolds, appeared in the 1913 paper [VA] by O. Veblen (1880–1960) and J. W. Alexander (1888–1971). The topological invariance of the Betti numbers and torsion coefficients of a manifold was established by Alexander in 1915. In 1923, Hermann Künneth (1892–1975) calculated the Betti numbers and torsion coefficients for a product of manifolds in [K23]; his results have since become known as the *Künneth Formulas.*

Until the mid 1920's, topologists studied homology via incidence matrices, which they could manipulate to determine the Betti numbers and torsion coefficients. This changed in 1925, when Emmy Noether (1882–1935) pointed out in her 14-line report [N25], and in her lectures in Göttingen, that homology was an *abelian group*, rather than just Betti numbers and torsion coefficients, and perceptions changed forever. The young H. Hopf (1894–1971), who had just arrived to spend a year in Göttingen and meet P. Alexandroff, realized how useful this viewpoint was, and the word spread rapidly. Inspired by the new viewpoint, the 1929 paper [M29] by L. Mayer (1887–1948) introduced the purely algebraic notions of chain complex, its subgroup of cycles and the homology groups of a complex. Slowly the subject became more algebraic.

During the decade 1925-1935 there was a general movement to extend the principal theorems of algebraic topology to more general spaces than those considered by Poincaré. This led to several versions of homology. Some people who invented homology theories in this decade were: Alexander [A26], Alexandroff (1896–1982), Čech (1893–1960) [C32], Lefschetz (1884–1972) [L33], Kolmogoroff (1903–1987), Kurosh (1908–1971) and Vietoris (1891–!). In 1940, Steenrod (1910–1989) developed a homology theory for compact metric spaces [S40], and his theory also belongs to this movement.

In each case, the homology theory could be described as follows: given topological data, the inventors gave an ad hoc recipe for constructing a chain complex, and defined their homology groups to be the homology of that chain complex. In each case, they showed that the result is independent of choices, and provides the usual Betti numbers for compact manifolds. One theme in many recipes was homology with coefficients in a compact topological group; this kind of homology remained in vogue until the early 1950's, by which time it had become superfluous. We shall pass over most of this decade, as it played little part in the development of homological algebra per se.

One theory we should mention is the "de Rham homology" of a smooth manifold, which was introduced by G. de Rham (1903–1990) in his 1931 thesis [dR]. Elie Cartan (1869–1951) had just introduced the cochain complex of exterior differential forms on a smooth manifold M in a series of papers [C28, C29] and had conjectured that the Betti number b_i of M is the maximum number of closed i-forms ω_j such that no nonzero linear combination $\sum \lambda_j \omega_j$ is exact. When de Rham saw Cartan's note [C28] in 1929, he quickly realized that he could solve Cartan's conjecture using a triangulation on M and the bilinear map

$$(C, \omega) \mapsto \int_C \omega.$$

Here C is an i-cycle for the triangulation and ω is a closed i-form. Indeed, Stokes' formula shows that $\int_C \omega = 0$ if either ω is an exact form or if C is a boundary. De Rham showed the converse was true: if we fix C then $\int_C \omega = 0$ if and only if C is a boundary, while if we fix ω then $\int_C \omega = 0$ if and only if ω is exact. De Rham's theorem proves Cartan's conjecture, since if we write $H^i_{dR}(M)$ for the quotient of all closed forms by the exact forms, then it gives a nondegenerate pairing between the vector spaces $H_i(M;\mathbb{R})$ and $H^i_{dR}(M)$.

Of course, $H^i_{dR}(M)$ is just the ith cohomology of Cartan's complex, and we now refer to it as the "de Rham cohomology" of M. But cohomology had not been invented in 1931, and no one seems to have realized this fact until Cartan and Chevalley in the 1940's, so de Rham was forced to state his results in terms of homology. Much later, the de Rham cohomology of Lie groups would then play a critical role in the development of the cohomology of Lie algebras (see [ChE] and the discussion below).

The rise of algebraic methods (1935–1945).

The year 1935 was a watershed year for topology in many ways. We shall focus upon four developments of importance to homology theory.

The *Hurewicz maps* $h\colon \pi_n(X) \to H_n(X;\mathbb{Z})$ were constructed and studied by Witold Hurewicz (1904–1956) in 1935. Hurewicz also studied *aspherical* spaces, meaning spaces such that $\pi_n(X) = 0$ for $n \neq 1$. He noticed in [Hu36] that if X and X' are two finite dimensional aspherical spaces with $\pi_1(X) = \pi_1(X')$ then X and X' are homotopy equivalent. From this he concluded that the homology $H_n(X;\mathbb{Z})$ of such an X depended only upon its fundamental group $\pi_1(X)$. This observation forms the implicit definition of the cohomology of a group, a definition only made explicit a decade later (see below).

The homology of the classical Lie groups was calculated in 1935 by Pontrjagin [P35] (Betti numbers only, using combinatorial proofs) and more fully by R. Brauer [B35] (ring structure, using de Rham homology). These calculations led directly to the modern study of Hopf algebras, as follows. H. Hopf introduced H-spaces in the paper [Hf41], written in 1939, and showed that the Brauer-Pontrjagin calculations were a consequence of the fact that the homology ring $H_*(M;\mathbb{Q})$ of any H-space M is an exterior algebra on odd generators; today we would say that Hopf's result amounted to an early classification of finite-dimensional graded "Hopf algebras" over $\mathbb{Q}$.

The third major advance was the determination of Universal Coefficient groups for homology, that is, a coefficient group A_u which would determine the homology groups $H_*(X;A)$ for arbitrary coefficients A. For finite complexes, where matrix methods apply, J. W. Alexander had already shown in 1926 [A26] that $H_*(X;\mathbb{Z}/n)$ was determined by $H_*(X;\mathbb{Z})$, the case $n = 2$ having been done as early as 1912 [VA]. In the 1935 paper [C35], E. Čech discovered that $\mathbb{Z}$ is a Universal Coefficient group for homology: assume that there is a chain complex C_* of free abelian groups, whose homology gives the integral homology of a space X (the space is introduced only for psychological reasons). Then for every abelian group A and every complex X, $H_n(X;A)$ is the direct sum of two subgroups, determined explicitly by $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$, respectively.

In fact, Čech's Universal Coefficient Theorem gave explicit presentations for these subgroups, which today we would recognise as presentations for $H_n(X;\mathbb{Z}) \otimes A$ and $\mathrm{Tor}_1(H_{n-1}(X;\mathbb{Z}), A)$. Thus Čech was the first to introduce the general tensor product and torsion product Tor of abelian groups into homological algebra. However, such a modern formulation of Čech's result (and the name Tor, due to Eilenberg around 1950) did not appear in print before 1951 (Exposé 10 of [C50]; see also p. 161 of [ES]). We note a contemporary variant in passing: Steenrod proved a Universal Coefficient Theorem for cohomology with coefficients in a compact topological group; see [S36]; in this context the Universal Coefficient group is the character group $\mathbb{R}/\mathbb{Z}$ of $\mathbb{Z}$.

The fourth great advance in 1935 was the discovery of cohomology theory and cup products, simultaneously and independently by Alexander and Kolmogoroff. The drama of their back-to-back presentations at the Moscow International Conference on Topology in September 1935 is nicely described in Massey's article [MHC] in this book. The Alexander-Kolmogoroff formulas defining the cup product were completely ad hoc, and also not exactly correct; the rectification was quickly discovered by Čech and Hassler Whitney (1907–1989), and corrected by Alexander. All three authors published articles about the cup product in the Annals of Mathematics during 1936–1938. Whitney's article [W37] had the most enduring impact, for it introduced the modern "co" terminology: *coboundary* (δ) and *cocycle*, as well as the notation $a \smile b$ and $a \frown b$, prophetically suggesting that "we might call $\smile$ 'cup' and $\frown$ 'cap'." Whitney's article also implicitly introduced the notion of what we now call a differential graded algebra, via the "Leibniz axiom" that if a and b are homogeneous of degrees p and q then:

$$\delta(a \smile b) = (\delta a) \smile b + (-1)^p a \smile \delta b.$$

During the next decade, while the world was at war, the algebraic machinery slowly fell into place.

In the 1938 paper [W38], Hassler Whitney discovered the tensor product construction $A \otimes B$ for abelian groups (and modules). Up to that time, this operation had only been known (indirectly) in special cases: the tensor product of vector spaces, or the tensor product of A with a finitely generated abelian group B. Whitney took the name from the following classical example in differential geometry: if T is the tangent vector space of a manifold at a point, then $T \otimes T$ is the vector space of (covariant) "tensors of order 2." The full modern definition of the tensor product (using left and right modules) appeared in Bourbaki's influential 1943 treatment [B43], as well as in the 1944 book [ANT] by Artin, Nesbitt and Thrall.

The concept of an exact sequence first appeared in Hurewicz' short abstract [Hu41] of a talk in 1941. This abstract discusses the long exact sequence in cohomology associated to a closed subset $Y \subset X$, in which the operation $\delta\colon H^q(X - Y) \to H^{q+1}(X, Y)$ plays a key role.

In the 1942 paper [EM42], Eilenberg (1915–) and Mac Lane (1909–) gave a treatment of the Universal Coefficient Problem for cohomology, naming Hom and Ext for the first time. Using these, they showed that Čech homology with coefficients in any abelian group A is determined by Čech cohomology with coefficients in $\mathbb{Z}$. This application further established the importance of algebra in topology. We will say more about this discovery in the next section.

In 1944, S. Eilenberg defined singular homology and cohomology in [E44]. First, he introduced the singular chain complex $S(X)$ of a topological space, and then he defined $H_*(X;A)$ and $H^*(X;A)$ to be the homology and cohomology of the chain complexes $S(X) \otimes A$ and $\mathrm{Hom}(S(X), A)$, respectively. The algebra of chain complexes was now firmly entrenched in topology. Eilenberg's definition of $S(X)$ was only a minor modification of Lefschetz' construction in [L33], replacing the notion of oriented simplices by the use of simplices with ordered vertices; this trick avoided the issue of equivalence relations on oriented simplices which introduced "degenerate" chains of order 2. (See [MHC].)

We close our description of this era with the 1945 paper by Eilenberg and Steenrod's [ES45]. This paper outlined an axiomatic treatment of homology theory, rederiving the whole of homology theory for finite complexes from these axioms. They also pointed out that singular homology and Čech homology satisfy the axioms, so they must agree on all finite complexes. The now-familiar axioms introduced in this paper were: functoriality of H_q and ∂; homotopy invariance; long exact homology sequence for $Y \subset X$; excision; and the *dimension axiom*: if P is a point then $H_q(P) = 0$ for $q \neq 0$. We refer the reader to the article [May] for subsequent developments on generalized homology theories, which are characterized by the Eilenberg-Steenrod axioms with the dimension axiom replaced by Milnor's *wedge axiom* [M62].

Homology and cohomology of algebraic systems

During the period 1940–1950, topologists gradually began to realize that the homology theory of topological spaces gave invariants of algebraic systems. This process began with the discovery that group extensions came up naturally in cohomology. Then came the discovery that the cohomology of an aspherical space Y and of a Lie group G only depended upon algebraic data: the fundamental group $\pi = \pi_1(Y)$ and the Lie algebra $\mathfrak{g}$ associated to G, respectively. This led to thinking of the homology and cohomology groups of Y and G as intrinsic to π and $\mathfrak{g}$, and therefore algebraically defineable in terms of the group π and the Lie algebra $\mathfrak{g}$.

Ext **of abelian groups.**

If A and B are abelian groups, an *extension* of B by A is an abelian group E, containing B as a subgroup, together with an identification of A with E/B. The set $\mathrm{Ext}(A,B)$ of (equivalence classes of) extensions appeared as a purely algebraic object, as a special case of the more general problem of group extensions (see below), decades before it played a crucial role in the development of homological algebra.

Here is the approach used by Reinhold Baer in 1934 [B34]. Suppose that we fix a presentation of an abelian group A by generators and relations: write $A = F/R$, where F is a free abelian group, say with generators $\{e_i\}$, and R is the subgroup of relations. If E is any extension of B by A, then by lifting the generators of A to elements $a(e_i)$ of E we get an element $a(r)$ of B for every relation r in R. Brauer thought of this as a function from the defining relations of A into B, so he called the induced homomorphism $a\colon R \to B$ a *relations function*. Conversely, he observed that every relations function a gives rise to a factor set, and hence to an extension $E(a)$, showing that two relations functions a and a' gave the same extensions if and only if there are elements b_i (corresponding to a function $b\colon F \to B$) so that

$a'(r) = b(r) + a(r)$ [B34, p. 394]. Finally, Baer observed (p. 395) that the formal sum $a + a'$ of two relations functions defined an addition law on $\mathrm{Ext}(A, B)$, making it into an abelian group. In his honor, we now call the extension $E(a + a')$ the "Baer sum" of the extensions.

Baer's presentation $A = F/R$ amounted to a free resolution of A, and his formulas were equivalent to the modern calculation of $\mathrm{Ext}(A, B)$ as the cokernel of $\mathrm{Hom}(F, B) \to \mathrm{Hom}(R, B)$. But working with free resolutions was still a decade away ([Hf44, F46]), and using them to calculate $\mathrm{Ext}(A, B)$ was even further in the future ([ES]).

We now turn to 1941. That year, Saunders Mac Lane gave a series of lectures on group extensions at the University of Michigan. According to [M88], most of the lectures concerned applications to Galois groups and class field theory, but Mac Lane ended with a calculation of the abelian extensions of $\mathbb{Z}$ by $A = \mathbb{Z}[\frac{1}{p}]$. Samuel Eilenberg, who had recently emmigrated from Poland and was an Instructor at Michigan, could not attend the last lecture and asked for a private lecture. Eilenberg immediately noticed that the group $\mathbb{Z}[\frac{1}{p}]$ was dual to the topological p-adic solenoid group Σ, which Eilenberg had been studying, and that Mac Lane's algebraic answer $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) = \hat{\mathbb{Z}}_p/\mathbb{Z}$ coincided with the Steenrod homology groups $H_1(S^3 - \Sigma; \mathbb{Z})$ calculated (by Steenrod) in [S40]. After an all-night session, followed by several months of puzzling over this observation, they figured out how Ext plays a role in cohomology; the result was the paper [EM42].

Time has recognized their result as the Universal Coefficient Theorem for singular cohomology, but singular cohomology had not yet been invented in 1942. In addition, the notation then in vogue, and used in [EM42], was the opposite of today's conventions (which date to [ES45] and [CE]) in several respects. They wrote $H^q(A)$ for the homology groups they worked with, and wrote $H_q(A)$ for the cohomology groups under consideration. And since they were reworking many of Baer's observations about extensions, they wrote $\mathrm{Ext}\{B, A\}$ for what we call $\mathrm{Ext}(A, B)$.

Here is a translation of their Universal Coefficient Theorem into modern language. Given an infinite but star-finite CW complex K, they formed the cochain complex $C^*(K)$ of *finite* cocycles with integer coefficients; each $C^q(K)$ is a free abelian group. Define the cohomology $\mathcal{H}^*$ of K using $C^*(K)$, and define the homology $H_*(K; A)$ of K with coefficients in A using the chain complex $\mathrm{Hom}(C^*(K), A)$. Then $H_q(K; A)$ is the product of $\mathrm{Hom}(\mathcal{H}^q, A)$ and the group $\mathrm{Ext}(\mathcal{H}^{q+1}, A)$ of abelian group extensions. Of course, the proof in [EM42] only uses the algebraic properties of $C^*(K)$. Since [ES] and [CE] it has been traditional to state this result the other way: given a chain complex C_* of free abelian groups, one sets $H_q = H_q(C_*)$ and describes the cohomology of the cochain complex $\mathrm{Hom}(C_*(K), A)$ as the product of $\mathrm{Hom}(H_q, A)$ and $\mathrm{Ext}(H_{q-1}, A)$.

In order to find a universal coefficient formula for the cohomology $H^q(K; A)$ of $C^*(K) \otimes A$, they discovered the "adjunction" isomorphism $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$; see [EM42, (18.3)]. This is an isomorphism which varies naturally with the abelian groups A, B and C. With this in hand, they reformulated Čech's Universal Coefficient Theorem: $H_q(K; A)$ is the direct sum of $H_q \otimes A$ and a group T that we would write as $T = \mathrm{Hom}_{\mathrm{cont}}(A^*, H_{q-1})$, where A^* is the Pontrjagin dual of A. In fact, T is $\mathrm{Tor}(A, H_{q-1})$; see [CE, p. 138].

The notion that $\mathrm{Hom}(A, B)$ varies naturally, contravariantly in A and covariantly in B, was central to the discussion in [EM42]. In order to have a precise language for speaking of this property for Hom, and for homology and cohomology, Eilenberg and Mac Lane concocted the notions of *functor* and *natural isomorphism* in 1942. They expanded the language to include *category* and *natural transformation* in 1945; see [EM45]. Although these concepts were used in several papers, the new language of Category Theory did not gain wide acceptance until the appearance of the books [ES] and [CE] in the 1950's.

Cohomology of Groups.

The low dimensional cohomology of a group π was classically studied in other guises, long before the notion of group cohomology was formulated in 1943–45. For example, $H^0(\pi; A) = A^G$, $H_1(\pi; \mathbb{Z}) = \pi/[\pi, \pi]$ and (for π finite) the character group $H^2(\pi, \mathbb{Z}) = H^1(\pi; \mathbb{C}^*) = \mathrm{Hom}(\pi, \mathbb{C}^*)$ were classical objects.

The group $H^1(\pi, A)$ of *crossed homomorphisms* of π into a representation A is just as classical: Hilbert's "Theorem 90" (1897) is actually the calculation that $H^1(\pi, L^\times) = 0$ when π is the Galois group of a cyclic field extension L/K, and the name comes from its role in the study of crossed product algebras [BN].

The study of $H^2(\pi; A)$, which classifies extensions over π with normal subgroup A via factor sets, is equally venerable. The idea of factor sets appeared as early as Hölder's 1893 paper [Hö, §18], again in Schur's 1904 study [S04] of projective representations $\pi \to PGL_n(\mathbb{C})$ (these determine an extension E over π with subgroup $\mathbb{C}^*$, equipped with an n-dimensional representation) and again in Dickson's 1906 construction of crossed product algebras. O. Schreier's 1926 paper [S26] was the first systematic treatment of factor sets; Schreier did not asssume that A was abelian. In 1928, R. Brauer used factor sets in [B28] to represent central simple algebras as crossed product algebras in relation to the Brauer group; this was clarified in [BN]. In 1934, R. Baer gave the first invariant treatment of extensions (i.e., without using factor sets) in [B34]. He noticed that when A was abelian, Schreier's factor sets could be added termwise, so that the extensions formed an abelian group. Extensions with A abelian were also studied by Marshall Hall in [H38].

The next step came in 1941, when Heinz Hopf submitted a surprising 2-page announcement [Hf42] to a topology conference at the University of Michigan. In it he showed that the fundamental group $\pi = \pi_1(X)$ determined the cokernel of the Hurewicz map $h\colon \pi_2(X) \to H_2(X; \mathbb{Z})$. If we present π as the quotient $\pi = F/R$ of a free group F by the subgroup R of relations, Hopf gave the explicit formula:

$$\frac{H_2(X; \mathbb{Z})}{h\big(\pi_2(X)\big)} \cong \frac{R \cap [F, F]}{[F, R]}.$$

In particular, if $\pi_2(X) = 0$ this shows exactly how $H_2(X; \mathbb{Z})$ depends only upon $\pi_1(X)$; this formula is now called *Hopf's formula* for $H_2(\pi; \mathbb{Z})$.

Communication with Switzerland was difficult during World War II, and Hopf's paper arrived too late to be presented at the conference, but his result made a big impression upon Eilenberg. According to Mac Lane [M88], Eilenberg suggested that they try to get rid of that non-invariant presentation of $\pi(X)$. Since they had just learned in [EM42] that homology determined cohomology, was it more efficient to describe the effect of $\pi_1(X)$ on $H^2(X;\mathbb{Z})$? Mac Lane states that this line of investigation provided the justification for the abstract study of the cohomology of groups, and "was the starting point of homological algebra" ([ML, p. 137]).

The actual definition of the homology and cohomology of a group π first appeared in the announcement [EM43] by Eilenberg and Mac Lane (the full paper appeared in 1945). At this time (March 1943 until 1945) Eilenberg and Mac Lane were housed together at Columbia, working on war-related applied mathematics [M89]. Independently in Amsterdam, Hans Freudenthal (1905-1990) discovered homology and cohomology of groups using free resolutions; his paper [F46] was probably smuggled out of the Netherlands in late 1944. Also working independently of Eilenberg-Mac Lane and Freudenthal, but in Switzerland, homology was defined in Hopf's paper [Hf44], and (based on Hopf's paper) the cohomology ring was defined in Beno Eckmann's 1945 paper [Eck]. We will discuss these approaches, beginning with [EM43].

Given π, Eilenberg and Mac Lane choose an aspherical space Y with $\pi = \pi_1(Y)$. Using Hurewicz' observation that the homology and cohomology groups of Y (with coefficients in A) were independent of the choice of Y, Eilenberg and Mac Lane took them as the definition of $H_n(\pi;A)$ and $H^n(\pi;A)$. To perform computations, Eilenberg and Mac Lane chose a specific abstract simplicial complex $K(\pi)$ for the aspherical space Y. Its n-cells correspond to ordered arrays $[x_1,\cdots,x_n]$ of elements in the group. Thus one way to calculate the cohomology groups of π was to use the cellular cochain complex of $K(\pi)$, whose n-chains are functions $f\colon \pi^q \to A$ from q copies of π to A. Eckmann's paper [Eck] also defines $H^q(X;A)$ as the cohomology of this ad hoc cochain complex, and defines the cohomology cup product in terms of this complex. Both papers showed that $H^2(G;A)$ classifies group extensions.

At the same time, Hopf gave a completely different definition in [Hf44]. First Hopf considers a module M over any ring R, and constructs a resolution F_* of M by free R-modules. If I is an ideal of R, he considers the homology of the kernel of $F_* \to F_*/I$ and shows that it is independent of the choice of resolution. In effect, this is the modern definition of the groups $\mathrm{Tor}_*^R(M,R/I)$! Hopf then specializes to the group ring $R = \mathbb{Z}[\pi]$, the augmentation ideal I and $M = \mathbb{Z}$, and defines the homology of π to be the result. That is, Hopf's definition is literally (in modern notation)

$$H_n(\pi;\mathbb{Z}) = \mathrm{Tor}_n^{\mathbb{Z}[\pi]}(\mathbb{Z},\mathbb{Z}).$$

Finally, Hopf showed that if Y is an aspherical cell space with $\pi = \pi_1(Y)$ then $H_n(Y;\mathbb{Z}) = H_n(\pi;\mathbb{Z})$. His proof has since become standard: the cellular chain complex F_* for the universal cover of Y is a free $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}$, and F_*/I is the cellular chain complex of Y. Thus the homology of F_*/I simultaneously computes the Betti homology of Y and the group homology of π, as claimed.

Freudenthal's method [F46] was similar to Hopf's, but less general. He considered a free $\mathbb{Z}[\pi]$-module resolution F_* of $\mathbb{Z}$, and showed that the homology of $F_* \otimes_\pi A$ is independent of F_* for every abelian group A. Like the others, Freudenthal constructed one such resolution from an aspherical polytope Y with $\pi = \pi_1(Y)$.

At first, calculations of group homology were restricted to those groups π which were fundamental groups of familiar topological spaces, using the bar complex. In his 1946 Harvard thesis [Lyn], R. Lyndon found a way to calculate the cohomology of a group π, given a normal subgroup N such that $H^*(N)$ and $H^*(\pi/N; A)$ were known. His procedure started with $H^p(\pi/N; H^q(N))$ and proceeded through successive subquotients, ending with graded groups associated to a filtration on $H^*(\pi)$. Serre quickly realized [S50] that Lyndon's procedure amounted to a spectral sequence, and completed the description with Hochschild in [HS53]. Since then, it has been known as the Lyndon-Hochschild-Serre spectral sequence.

One application of the new definitions was *Galois cohomology*, so named in Hochschild's study [Hh50] of local class field theory. If L is a finite Galois extension of a field K with Galois group G, this referred to the cohomology of G with coefficients in $L^\times$, or in a related G-module such as the idèle class group of L. For example, the Normal Basis Theorem implies that the additive group L is a free G-module over L, so $H^q(G; L) = 0$ for $q \neq 0$ [E49]. Early on, it was observed that the factor sets of Brauer [B28] and Brauer-Noether [BN] were 2-cocycles, and the Brauer-Noether results translated immediately into the following theorem about the Brauer group: $H^2(G; L^\times)$ is isomorphic to the kernel $Br(L/K)$ of the map $Br(K) \to Br(L)$, and is generated by the central simple algebras which are split over L. This observation was mentioned in Eilenberg's 1948 survey [E49] of the field. A careful writeup was given by Serre in Cartan's 1950/51 seminar [C50].

The 1952 paper [HN52] explored the connection to Class Field Theory, translating Tsen's Theorem (1933) into the vanishing of $H^q(G; K^\times)$ for $q \neq 0$ when k and K are function fields of curves over an algebraically closed field. This paper also marked the first appearance of *Shapiro's lemma*, a formula for the cohomology of an induced module which is due to Arnold Shapiro.

While studying Galois cohomology in his thesis [T52], John Tate discovered that there is a natural isomorphism $H^r(G; \mathbb{Z}) \cong H^{r+2}(G; C_L)$, where C_L is the idèle class group of a number field L. Moreover, the reciprocity law gave a similar relation between $H_1(G; \mathbb{Z}) = G/[G, G]$ and a subgroup of $H^0(G; C_L)$. This led him to define the *Tate cohomology* $\widehat{H}^*(G, A)$ of any finite group G and any G-module A, indexed by all integers; see [T54]. Tate did this by splicing together the cohomology of G ($\widehat{H}^r(G; A) = H^r(G; A)$ for $r > 0$) and the homology of G (reindexing via H_n as $\widehat{H}^{-n-1}$ for $n \geq 1$), and using ad hoc definitions for $\widehat{H}^0$ and $\widehat{H}^{-1}$.

The 1950/51 Seminaire Cartan [C50] saw the next major advances in group homology. In Exposés 1 and 2, Eilenberg gave an axiomatic characterization of homology and cohomology theories for a group π, and used a fixed free resolution of the π-module $\mathbb{Z}$ to establish the existence of both a homology and a cohomology theory. The key axioms Eilenberg introduced to prove uniqueness were: 1) if A is a free π-module then $H_q(\pi; A) = 0$ for $q > 0$, and 2) if A is an injective π-module then $H^q(\pi; A) = 0$ for $q > 0$. In Exposè 4 of the same seminar, H. Cartan proved what we now call the Comparison Theorem for chain complexes: given a free resolution C_* and an acyclic resolution C'_* of $\mathbb{Z}$, there is a chain map $C_* \to C'_*$ over $\mathbb{Z}$, unique up to chain homotopy. This made Eilenberg's construction natural in the choice of C_*, and allowed Cartan the freedom to construct cup products in group cohomology via resolutions.

After the 1950–51 Seminaire Cartan [C50], the germs of a complete reworking of the subject were in place. Cartan and Eilenberg began to collaborate on this reworking, not realizing that the resulting book [CE] would take five years to appear.

Associative algebras.

Before the cohomology theory of associative algebras was defined, the special cases of derivations and extensions had been studied. Derivations and inner derivations of algebras (associative ot not) over a field k were first studied systematically in 1937 by N. Jacobson [J37], who was especially interested in the connection to Galois theory over k when $char(k) \neq 0$.

Hochschild studied derivations of associative algebras and Lie algebras in the 1942 paper [Hh42]. He showed that every derivation of an associative algebra A is inner if and only if A is a *separable* algebra, meaning that not only is A semisimple, but the ℓ-algebra $A \otimes_k \ell$ is semisimple for every extension field $k \subseteq \ell$. In addition, he showed that if A is semisimple over a field of characteristic zero, M is an A-bimodule, and $f\colon A \otimes A \to M$ is a bilinear map satisfying the *factor set condition*:

$$a\,f(b,c) + f(a,bc) = f(a,b)\,c + f(ab,c),$$

then there is a linear map $e\colon A \to M$ so that $f(a,b) = a\,e(b) + e(a)b - e(ab)$.

Upon seeing the Eilenberg-Mac Lane treatment of the cohomology of groups in 1945, Hochschild observed ([Hh45]) that the same formulas gave a purely algebraic definition of the cohomology of an associative algebra A over a field, with coefficients in a bimodule M. The degree q part $C^q(A;M)$ of his ad hoc cochain complex is the vector space of multilinear maps from A to M, i.e., linear maps $A^{\otimes q} \to M$. For example, if $e\colon k \to M$ has $e(1) = m$ then $\delta(e)(a) = am - ma$ is an inner derivation, and a 1-cocycle is a map $f\colon A \to M$ such that $f(ab) = af(b) + f(a)b$. Thus the construction makes $H^1(A;M)$ into the quotient of all derivations by inner derivations, and the first of Hochschild's 1942 results becomes: $H^1(A;M)$ vanishes for every M if and only if A is a *separable* algebra. Hochschild also showed that $H^2(A;M)$ measures algebra extensions E of A by M, meaning that M is a square-zero ideal and $E/M \cong A$; a trivial extension is one in which the algebra map $E \to A$ splits. Since a 2-cocycle is just a map $f\colon A \otimes A \to M$ satisfying the factor set condition mentioned above, Hochschild's second 1942 result becomes: if A is semisimple then $H^2(A;M)$ vanishes for every M, and hence every nilpotent algebra extension of A must be split.

Lie algebras.

Since Elie Cartan's theorem [C29] that every connected Lie group is diffeomorphic to the product of a compact Lie group G and $\mathbb{R}^n$, the cohomology of Lie groups was reduced to that of compact Lie groups. We have seen how this was solved in 1935 by Brauer and Pontrjagin. Later, Cartan and de Rham observed that the de Rham cohomology $H^*_{\mathrm{dR}}(G;\mathbb{R})$ of G may be computed using left invariant differentials, and it was gradually noticed that the Lie algebra $\mathfrak{g}$ of left invariant vector fields (or tangent vectors at the origin of G) determines the cohomology of G.

Chevalley and Eilenberg were able to use this observation to define the cohomology of any Lie algebra in their 1948 paper [ChE]. After reviewing de Rham cohomology, they calculate that the (differential graded) algebra of left invariant differential forms on a Lie group G is isomorphic to the dual algebra $C^*(\mathfrak{g})$ of the exterior algebra $\wedge^*\mathfrak{g}$. Translating the de Rham differential into this context gave the differential $\delta\colon C^q(\mathfrak{g}) \to C^{q+1}(\mathfrak{g})$ defined by

$$(\delta\omega)(x_1,\dots,x_{q+1}) = \frac{1}{q+1}\sum(-1)^{k+l+1}\omega([x_k,x_l],\dots,\hat{x}_k,\dots,\hat{x}_l,\dots).$$

This makes $C^*(\mathfrak{g})$ into a differential graded algebra, and they define the cohomology ring $H^*_{\mathrm{Lie}}(\mathfrak{g};\mathbb{R})$ of the Lie algebra $\mathfrak{g}$ to be the cohomology of $C^*(\mathfrak{g})$. (They then state that in other characteristics one can and should omit the constant $\frac{1}{q+1}$.) Thus if G is compact and connected then their construction of Lie algebra cohomology has the isomorphism $H^*_{\mathrm{dR}}(G;\mathbb{R}) \cong H^*_{\mathrm{Lie}}(\mathfrak{g};\mathbb{R})$ as its birth certificate.

It is immediate that a 1-cocycle is a map $\mathfrak{g} \to \mathbb{R}$ vanishing on the subalgebra $[\mathfrak{g},\mathfrak{g}]$. Since there are no 1-coboundaries we see that $H^1_{\mathrm{Lie}}(\mathfrak{g};k)$ is the dual space of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. This purely algebraic feature is present, but had been downplayed in the cohomology of compact connected Lie groups, because it follows from the fact that $G/[G,G]$ is a torus.

In order to study the cohomology $H^*_{\mathrm{dR}}(G/H;\mathbb{R})$ of the homogeneous spaces G/H of G, Chevalley and Eilenberg also defined the cohomology $H^*_{\mathrm{Lie}}(\mathfrak{g};V)$ of a representation V of $\mathfrak{g}$. This was defined similarly, as the cohomology of the chain complex $C^*(\mathfrak{g};V)$ of (vector space) maps from $\wedge^*\mathfrak{g}$ to V. Translated from the corresponding de Rham differential on the manifold G/H, the formula for the differential $\delta\omega$ resembled the one displayed above, but it had an extra alternating sum of terms $x_k\omega(\cdots,\hat{x}_k,\cdots)$.

According to Jacobson [J37], a derivation from a Lie algebra $\mathfrak{g}$ into a $\mathfrak{g}$-module V is a linear map $D\colon \mathfrak{g} \to V$ such that $D([x,y]) = x(Dy) - y(Dx)$. It is an inner derivation if $D(x) = xv$ for some $v \in V$. It is immediate from the Chevalley-Eilenberg complex $C^*(\mathfrak{g};V)$ that $H^1_{\mathrm{Lie}}(\mathfrak{g};V)$ is the quotient of all derivations from $\mathfrak{g}$ into M by the inner derivations.

The paper [ChE] also contains the theorem that Lie extensions of $\mathfrak{g}$ by V are in one-to-one correspondence with elements of $H^2(\mathfrak{g};V)$, a result inspired by Eilenberg's role in the earlier classification of group extensions via $H^2(G;A)$ in [EM43]. Indeed, the proof was similar: cocycles in the complex $C^*(\mathfrak{g};V)$ are recognised as factor sets for extensions.

Now suppose that $\mathfrak{g}$ is any semisimple Lie algebra over a field k of characteristic zero. J. H. C. Whitehead (1904–1960) had discovered some algebraic lemmas about linear maps on $\mathfrak{g}$ in 1936–37 (see [W36]), in order to give a purely algebraic proof of Weyl's 1925 Theorem that every representation is completely reducible. Whitehead's lemmas also appeared in Hochschild's paper [Hh42] on derivations. Whitehead's "first lemma" said that every derivation from $\mathfrak{g}$ into any representation V was inner, even though he proved this result before the notion of derivation was known. Chevalley and Eilenberg translated Whitehead's "first lemma" as the statement that $H^1_{\mathrm{Lie}}(\mathfrak{g};V) = 0$ for all V.

Whitehead's "second lemma" concerned alternating bilinear maps $f\colon \mathfrak{g}\wedge\mathfrak{g} \to V$ satisfying a *factor set condition*, which we would now write as $\delta f(x,y,z) = 0$. Whitehead proved that for every such f there was always a linear map $e\colon \mathfrak{g} \to V$ so that $f(x,y) = x\,e(y) - y\,e(x) + e([x,y])$. Chevalley and Eilenberg translated this as the statement that $H^2_{\mathrm{Lie}}(\mathfrak{g};V) = 0$ for all V. In both of these results, the first step was an analysis of the trivial representation $V = k$. For the second step, they used another result of Whitehead to show that when $V \neq k$ is a simple representation then $H^q_{\mathrm{Lie}}(\mathfrak{g}; V) = 0$ for all q. This last step shows that the only interesting cohomology groups of $\mathfrak{g}$ are those with trivial coefficients, and these are interesting because $H^q_{\mathrm{Lie}}(\mathfrak{g}; k) = H^q(G; k)$.

The analogy with the cohomology of compact Lie groups was pursued further by Koszul (1921–) in [K50]. He introduced the notion of a reductive Lie algebra $\mathfrak{g}$, and showed that (in characteristic zero) its cohomology is an exterior algebra.

Sheaves and Spectral Sequences.

Jean Leray (1906–) was a prisoner of war during World War II, from 1940 until 1945. He organized a university in his prison camp and taught a course on algebraic topology. At the end of his imprisonment, he invented *sheaves* and *sheaf cohomology* [L46a], as well as *spectral sequences* for computing his sheaf cohomology [L46b].

As we saw above, the essential features of a spectral sequence had also been noted independently by R. Lyndon [Lyn], as a way to calculate the cohomology of a group. The algebraic properties of spectral sequences were codified by Koszul [K47] in 1947, using Cartan's suggestion that the central object should be a filtered chain complex. Koszul's presentation clarified things so much that Leray immediately adopted Koszul's framework.

In 1947–48, Leray gave a course at the College de France on this new cohomology theory. Part I was a review of his theory of spectral sequences, using Koszul's framework. Part II introduced the notion of a sheaf, and the cohomology of a locally compact topological space relative to a differential graded sheaf. The details of this course eventually appeared in Leray's detailed article [L50].

The next year (1948-49), Henri Cartan ran a Seminar [C48] on algebraic topology, with 17 exposés published as unbound mimeographed notes. Exposés XII–XVII were devoted to an exposition of Leray's theory of sheaves, but were withdrawn when Cartan's viewpoint on sheaves changed later that year. The same subject was revisited by H. Cartan two years later in Exposés 14–20 of the 1950-51 Cartan Seminar [C50], where he and his students reworked the theory of sheaves, and sheaf cohomology, based on the notion of a "fine" sheaf.

In Exposé 16 Cartan gave axioms for sheaf cohomology theory on a paracompact space X (with or without supports in a family Φ of closed subspaces of X, which we shall omit from our notation here). His axioms were: $H^0(X, F)$ is the group $\Gamma(F)$ of global sections of the sheaf F (with support in Φ); $H^q(X, F)$ depends functorially on F and vanishes for negative q; a natural long exact cohomology sequence exists for each short exact sequence of sheaves; and if F is a "fine" sheaf then $H^q(X, F) = 0$ for all $q \neq 0$.

Cartan was now able to mimic the proof of existence and uniqueness for group cohomology given earlier in Exposés 1–4 of the same Seminar by Cartan and Eilenberg. To prove uniqueness, he observed that every sheaf F may be embedded in a fine sheaf, specifically into a sheaf he called $F \otimes S$, which we would describe as the sum of the skyscraper sheaves $x_* x^*(F)$ over all points x of X. To prove existence, Cartan fixed a resolution $0 \to \mathbb{Z} \to C_0 \to \cdots$ of $\mathbb{Z}$ by torsionfree fine sheaves, and set $H^q(X, F) = H^q(\Gamma(C \otimes F))$. Observing that some choices of C happen to give differential graded algebras, he was able to define a product structure

$H^p(X, F) \otimes H^q(X, F') \to H^{p+q}(X, F \otimes F')$ on sheaf cohomology.

In the remaining exposés of [C50], Cartan, Eilenberg and Serre returned to Leray's spectral sequences, codifying the machinery and studying its multiplicative structure. Much of this material was reproduced in the Hochschild-Serre paper [HS53] in order to redo Lyndon's spectral sequence [Lyn]. The usefulness of this approach to spectral sequences was decisively demonstrated by Serre in his 1951 thesis [Se51].

A completely different approach to spectral sequences was given by W. Massey in 1952 ([M52]). Massey defines an *exact couple* to be a pair of (graded) modules D and E, equipped with maps fitting into an exact sequence

$$D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{k} D \xrightarrow{i} D.$$

One forms its *derived couple* by considering $D_1 = i(D)$ and the homology E_1 of E with respect to the differential jk. By an iterative process, one obtains a sequence of derived couples, and the sequence of modules E_r forms a spectral sequence. The exact couple approach to spectral sequences has since become very popular with topologists, but less so with algebraists.

Godement's 1958 book [Gode] summarized and refined all these developments, becoming the standard reference for sheaves, sheaf cohomology and spectral sequences for many years. In Godement's approach, the focus moved away from Cartan's notion of "fine" sheaf and towards the new notions of *flasque* and *soft (mou)* sheaves. One trick introduced by Godement, but implicit in Cartan's 1950–51 seminar [C50], was that by iteration of the canonical embedding of F into $F \otimes S$ one could get a resolution of F by injective sheaves which is functorial in F; nowadays it is called the *Godement resolution* of F.

The Cartan–Eilenberg Revolution

As we have mentioned, Cartan and Eilenberg began collaborating during the 1950–1951 Seminaire Cartan [C50], rewriting the foundations of all the ad hoc algebraic homology and cohomology theories that had arisen in the previous decade. Coining the term *Homological Algebra* for this newly unified subject, and using it for the title of the textbook [CE], they revolutionized the subject.

The first occurrence of the notation Tor_n and Ext^n, as well as the concepts of projective module, derived functor and hyperhomology appeared in this book. In his review of their book, Hochschild stated that "The appearance of this book must mean that the experimental phase of homological algebra is now surpassed."

Before we describe the innovations in their book further, let us back up and review the evolution of the two main tools that were now available, namely chain complexes and resolutions.

Chain Complexes.

The algebra of chain complexes had been slowly evolving since their formal introduction in 1929 by Mayer [M29]. We have already mentioned Hurewicz' 1941 discovery of the notion of exact sequence ([H41]), and the application of this notion in the 1945 axiomatization of homology theory [ES45].

The next step was taken in 1947 by Kelley and Pitcher [KP], who coined the term "exact sequence" and first systematically studied chain complexes from an algebraic point of view. They showed that direct limits preserve exact sequences (axiom AB5 holds), but that inverse limits do not (axiom AB5* fails). If A_* is a subcomplex of B_*, with quotient C_*, they constructed the boundary map $\partial\colon H_q(C) \to H_{q-1}(A)$ and proved that the long homology sequence

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is exact. Since they restricted themselves to positive complexes (indexed by positive integers q), their sequence ended in $H_0(B) \to H_0(C) \to 0$.

The yoga of chain complexes was further developed in Eilenberg and Steenrod's book [ES]; cf. [ES45]. They indexed their chain complexes by all integers, and observed that cochain complexes could be identified as chain complexes via the reindexing $C_q = C^{-q}$. The familiar "five-lemma" occurs for the first time on p. 16 of [ES]. (Its companion, the "snake lemma," first appeared in [CE].) Eilenberg and Steenrod's book also introduced the "Mayer-Vietoris" sequence for a space $X = U \cup V$, associated to the excision isomorphism $H_*(U, U \cap V) \cong H_*(X, V)$.

Free and Injective resolutions.

Free resolutions have long been used in algebra, starting with David Hilbert (1862–1943) in his 1890 paper [Hilb] on iterated syzygies of a finitely generated graded module M over a polynomial ring $R = k[x_1, \ldots, x_n]$. A choice of $b_0 = \dim(M \otimes_R k)$ homogeneous generators of M defines a surjection $R^{b_0} \to M$, and its kernel is the first *syzygy module* of M. (There is a grading on R^{b_0} which we are ignoring.) Hilbert proved that the syzygy was also finitely generated (the Hilbert Basis Theorem), so one could use induction to define the higher syzygy modules. Hilbert's Syzygy Theorem states that the $(n+1)$st syzygy is always zero, i.e., the nth syzygy is R^{b_n} for some b_n. Since the number of generators b_i of the syzygies is chosen minimally, they are independent of the choices of generators: today we know this is so because b_i is the dimension of the vector space $\mathrm{Tor}_i^R(M, k)$. By analogy with topology, the b_i are called the *Betti numbers* of M.

As we have remarked, Baer [B34] implicitly used free resolutions of an abelian group to study the groups $\mathrm{Ext}(A, B)$. The next explicit use of free resolutions was by Hopf in 1944 [Hf44]. As we have mentioned above, he used them to describe the homology of a group, and implicitly gave a definition of the modules $\mathrm{Tor}_i^R(M, R/I)$ for any ideal I of any ring R. Based on Hopf's work, Cartan and Eilenberg used free $\mathbb{Z}[\pi]$-resolutions of a π-module A in [C50] to give an axiomatic description for the group homology $H_*(G; A)$.

Injective R-modules were introduced and studied in 1940 by R. Baer [B40]. Baer called them "complete" abelian groups over the ring R; the name *injective* apparently first arose in Eilenberg's survey paper [C50]. Baer's paper contains the proposition that every module is a submodule of an injective module, and what is now called "Baer's criterion" for M to be injective: every map from an ideal into M must extend to a map from R into M. Finally, Baer characterized semisimple rings as those for which every module is injective.

In the 1948 paper [M48], Mac Lane formulated the projective and injective lifting properties for the category of abelian groups, and showed that these properties describe free and divisible abelian groups, respectively. He did not discover the notion of projective module because he did not apply these lifting properties to categories of modules. Using this, he showed that one could compute $\mathrm{Ext}(A, B)$ by embedding the abelian group B in a divisible group D; this amounts to the use of an injective resolution of B.

Cartan and Eilenberg: the book.

We now turn to the contents of the book [CE] itself. On p. 6 it introduced an entirely new concept: the definition of a *projective* module. It proved on p. 11 that every R-module is projective if and only if R is semisimple, complementing Baer's characterization of semisimplicity in terms of injective modules; later in the book (p. 111), this was viewed as the characterization of rings of global dimension 0.

In chapter II the authors introduced the notion of left exact functors (such as Hom) and right exact functors (such as $\otimes_R$). In the central chapter V, they introduced the notions of *projective resolutions* $\cdots \to P_0 \to M$ and *injective resolutions* $M \to I^0 \to \cdots$ of a module M, and used these to define the derived functors $L_nT(M) = H_nT(P_*)$ and $R^nT(M) = H^nT(I^*)$ of an additive functor T. This material was clearly based on the ideas in the 1950–1951 Seminaire Cartan [C50].

In chapter VI, the authors defined $\mathrm{Tor}_n^R(M, N)$ and $\mathrm{Ext}_R^n(M, N)$ as the derived functors of $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$. Then they defined the projective and injective dimension of M as the length of the shortest projective and injective resolution, and characterized these dimensions in terms of the vanishing of $\mathrm{Ext}_R^n(M, -)$ and $\mathrm{Ext}_R^n(-, M)$, respectively. This led them to define the (left and right) *global dimension* of R as the largest n such that Ext_R^n is nonzero, and the *weak global dimension* (now called the Tor-*dimension*) as the largest n such that Tor_n^R is nonzero.

Chapters VIII to XIII unified the homology of augmented algebras, Hochschild's homology and cohomology of an associative algebra Λ (as Tor and Ext groups over the enveloping algebra $\Lambda \otimes \Lambda^{op}$), the homology and cohomology of a group π (as Tor and Ext groups over the group ring $\mathbb{Z}[\pi]$), and the homology and cohomology of a Lie algebra $\mathfrak{g}$ (as Tor and Ext groups over the enveloping algebra $U\mathfrak{g}$).

Chapters XV–XVI contained a very readable introduction to spectral sequences for filtered chain complexes, and applications to computing Ext and Tor. Again, this material was based on the ideas in the 1950–1951 Seminaire Cartan [C50].

The final Chapter (XVII) concerned the hyperhomology of a functor T applied to a chain complex A. This was the precursor to the discovery of the Derived Category by Grothendieck and Verdier [V]. First they defined double complexes they called "projective" and "injective" resolutions of A; since [HRD] we call them *Cartan-Eilenberg resolutions* of A. Then they defined the hyperhomology $\mathbb{L}_*T(A)$ and hypercohomology $\mathbb{R}^*T(A)$ to be the (co)homology of the total complex of T applied to the double complex resolutions.

Until 1970, [CE] was the bible on homological algebra, although Mac Lane's book [ML] was also popular. These texts helped the subject become standard course material. Grothendieck's Tohoku paper [G57], which we shall describe below, and later his multi-volume tome [EGA] on the foundation of sheaf cohomology in Algebraic Geometry, were also heavy favorites. A second generation of texts appeared in 1970–71: Rotman's *Notes on Homological Algebra* [Rot] and Hilton and Stammbach's book [HStm].

Abelian Categories.

As soon as Cartan and Eilenberg began their undertaking, limiting theselves to functors defined on modules, it was clear that there was more than a formal analogy with the cohomology of sheaves, and that their methods worked in a more general setting. The search for that setting led to the notion of an abelian category.

The first attempt to formulate a setting in which homological algebra made sense was by Mac Lane in 1948 [M48]. In this paper Mac Lane introduced what he called "abelian categories," but which were actually additive categories with special objects resembling the objects $\mathbb{Z}$ and $\mathbb{Q}/Z$ in the category **Ab** of abelian groups. The category of abelian semigroups was an abelian category in Mac Lane's sense. This notion never caught on, though.

The appendix to [CE] contained the next attempt, by D. Buchsbaum. It was actually a summary without proofs of his 1955 thesis [B55], written under Eilenberg. In attempting to formulate a general setting in which the theory in Cartan-Eilenberg could be generalized, he needed categories which had a natural notion of an exact sequence. To this end, Buchsbaum introduced the notion of an *exact category*, which is an abelian category without the requirement that direct sums exist. To handle functors of more than one variable, he introduced the extra axiom (V) that direct sums $A \oplus B$ exist, which is equivalent to the definition of an abelian category. Buchsbaum also introduced axioms that the category has *enough projectives* or *enough injectives*. These axioms, unnecessary for the categories of modules considered in [CE], allowed Buchsbaum to carry over verbatim the Cartan-Eilenberg construction of derived functors to exact categories.

The name *abelian category* is due to A. Grothendieck [G57] and A. Heller [H58]. Grothendieck's paper was motivated by the observation that the category $\mathbf{Sh}(X)$ of sheaves of abelian groups on a topological space X was an abelian category with enough injectives, so that sheaf cohomology could be defined as the right derived functors of the global sections functor, while Heller was more concerned with a formal analogy to stable homotopy (where syzygy modules correspond to loop spaces, and projective modules correspond to contractible spaces).

Grothendieck's 1957 "Tohoku" paper [G57] introduced a hierarchy of axioms (AB3)–(AB6) and (AB3*)–(AB6*) that an abelian category may or may not satisfy. Axioms (AB3) and (AB3*) specify that set-indexed coproducts and products exist, respectively. The abelian category $\mathbf{Sh}(X)$ satisfies axiom (AB5), that filtered colimits of exact sequences are exact, but not axiom (AB4*), which states that a product of surjections is a surjection.

Given this framework, Grothendieck proceeded to generalize Cartan and Eilenberg's treatment of derived functors, introducing the names ∂-functor and *universal* ∂-functor, as well as the notion of T-acyclic objects (in [CE, p. 122] flat modules were defined as Tor-acyclic modules; Grothendieck showed that Godement's flasque sheaves were Γ-acyclic sheaves). The primary computational tool introduced by Grothendieck was a special case of the hypercohomology spectral sequence for the composition TU of two functors (see the last page of [CE]). Grothendieck observed that if T and U were left exact, and if U sends injective modules to T-acyclic modules then we could write the spectral sequence as

$$(R^pT)(R^qU) \Longrightarrow R^{p+q}(TU).$$

Several of the spectral sequences in [CE] were seen to be simple special cases of Grothendieck's spectral sequence, but so were the Leray spectral sequences associated to a continuous map $f\colon Y \to X$ and a sheaf F on Y:

$$H^p(X, R^q f_* F) \Longrightarrow H^{p+q}(Y, F).$$

Even the simplest of lemmas (such as the Snake Lemma) were painfully difficult to prove in a general abelian category, because one couldn't chase elements that didn't exist. This technique of diagram-chasing was justified in 1960, when Saul Lubkin [L60], A. P. Heron (1960 Oxford thesis) and J. P. Freyd (1960 Princeton thesis) proved that every small abelian category admits an exact embedding into the category of abelian groups. Shortly thereafter, Freyd and Barry Mitchell proved a stronger version: every small abelian category admits a full exact embedding into the category of modules over some ring (see [F64]). With this result, and P. Gabriel's thesis [G62], the subject was near maturity.

After the Cartan–Eilenberg Revolution

Upon the publication of Cartan-Eilenberg [CE], there was an explosion of research in homological algebra. Some results appeared to be fairly isolated curiousities at the time, but later became important, such as Yoneda's definition of Ext^n groups by long exact sequences in [Y54], the 1961 study of $\lim^1$ by J. E. Roos [R61], the Eilenberg-Moore paper [EM62] on spectral sequences for complete filtered complexes, Giraud's work [G65] on nonabelian H^1 in a Grothendieck Topos, or Boardman's influential preprint [B81] on conditional convergence in spectral sequences. In this article we shall focus upon the strands of thought that have led to flourishing new fields of study.

Projective Modules.

When the notion of projective module was introduced in [CE], there were not many examples of projective modules which were not free. By [CE, p. 157], all finitely generated projective modules over a local ring are free. By [CE, p. 13], all projective modules over a principal ideal domain (or more generally a Bezóut domain) are free. Kaplansky later showed [K58] that *all* projective modules over a local ring are free, as a consequence of the general result that any infinitely generated projective module is a direct sum of countably generated projective modules.

If I is an ideal of an integral domain R, Cartan and Eilenberg showed that I was projective if and only if it was *invertible*: $I \cdot I^{-1} = R$. Moreover, if $\dim(R) = 1$ then invertible ideals have at most two generators, so $I \oplus I^{-1} \cong R \oplus R$. Since every ideal in a Dedekind domain is invertible — their isomorphism classes forming the *Picard class group* of R — and the integers in a number ring were Dedekind domains whose class groups were classical objects of study, some examples of non-free projective modules were already known in the late 19th century.

For some rings, it was possible to classify all projective modules. A *Prüfer domain* is a commutative domain in which every finitely generated ideal is invertible; this generalization of Dedekind domains is named for H. Prüfer, who initiated their study in 1923. Kaplansky [K52] showed that if R is a Prüfer domain then every finitely generated torsionfree module — hence every projective module — is a direct sum of invertible ideals; see [CE, pp. 13, 133].

For other rings, the classification was much harder. In Serre's classic 1955 paper [Se55, p.243], he stated that it was unknown whether or not every projective R-module was free when R is a polynomial ring over a field. This became known as the "Serre problem," and was not solved (affirmatively) until 1976, by Quillen [Q76] and Suslin [S76].

In the period 1958–1962 there was a flurry of examples of non-free projective modules, coming from algebraic geometry [BS, Se58], arithmetic [B61], group rings [Sw59] and topological vector bundles [Sw62]. Much of this was based upon the dictionary in Serre's 1955 paper [Se55], between projective modules and topological vector bundles. Grothendieck's Riemann-Roch Theorem, published in [BS], showed that the "projective class group" $K(R)$ of stable isomorphism classes of projective modules was useful, especially for rings coming from algebra and algebraic geometry. Bass, Serre and Swan began a study of the projective class group $K(R)$; by 1964 it was renamed $K_0(R)$ in view of its parallels to topological K-theory, and this led to the rise of algebraic K-theory in the 1960's.

Homological Algebra and ring theory.

The left and right global dimension of a ring were early targets. In [A55], M. Auslander (1926–1994) showed that the left and right global dimension of a noetherian ring agree, and equal the weak global dimension. Then M. Harada [H56] showed that the rings with weak global dimension 0 are precisely the von Neumann regular rings, so the weak dimension and global dimension need not agree. Examples in which the left and right global dimensions of a ring are different were not known until a decade later, and were found by Osofsky [O68].

Regular local rings.

A *regular* local ring is a commutative noetherian local ring R whose maximal ideal $\mathfrak{m}$ is generated by a regular sequence, or equivalently, such that $\dim(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$. Regular local rings had become important in algebraic geometry because they were the local coordinate rings of smooth algebraic varieties. Auslander and Buchsbaum [AB56] and Serre [Se56] used homological methods to characterize regular local rings as those (noetherian) local rings R with finite global dimension. If R is local with residue field k and $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$, Serre proved that $\mathrm{Tor}_n^R(k,k) \neq 0$. Hence $gl.\dim(R) \geq n$, and $n \geq \dim(R) \geq \mathrm{depth}(R)$. Auslander and Buchsbaum proved that the depth of R is an upper bound for the finite values of $pd_R(M)$, so if pd_R is always finite we must have equality: $gl.\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$. In particular, if $gl.\dim(R) < \infty$ then R must be regular.

Since localization cannot increase global dimension, a corollary is that any localization of a regular local ring is again a regular ring. This non-homological statement, proven by homological methods, firmly established homological algebra as a central tool in ring theory; the alternate non-homological proof of this localization result, due to Nagata [N58], is very long and hard.

Also in [AB56], Auslander and Buchsbaum proved that 2-dimensional regular local rings are Unique Factorization Domains (UFD's). A few years later, Auslander and Buchsbaum [AB59] used similar homological methods to prove that every regular local ring is a Unique Factorization Domain.

Two timely courses on this material, by Serre in France and Kaplansky in the U.S., had a lasting impact upon the field.

In 1957–58, Serre taught a course on multiplicities at the Collège de France [SeM]. Part of that course focussed upon the simple inequality $pd_R(M) \leq pd_R(S)+pd_S(M)$ for a module M over an R-algebra S (an exercise on p. 360 of [CE].) Auslander and Buchsbaum realized that Serre's methods could be used to study the connection between the codimension and multiplicity over a local ring; see [AB58]. This led them to the *Auslander-Buchsbaum Equality*: if M is a finitely generated module over a local ring R and $pd_R(M) < \infty$ then $depth(R) = depth(M) + pd_R(M)$.

In Fall 1958, Kaplansky taught a course [K59] on homological algebra at the University of Chicago. Several students attending this course would later make important contributions to the subject: H. Bass, S. Chase, E. Matlis and S. Shanuel.

Kaplansky's course was organized around three "change of rings" theorems, describing how homological dimension changes when one passes from a ring R to a quotient ring $R/(x)$. They allowed him to prove the Theorems of Serre and Auslander-Buchsbaum without having to first develop Ext or Tor. Early in the course, Shanuel noticed that there was an elegant relation between different projective resolutions of the same module. Kaplansky seized upon this result as a way to define projective dimension, and christened it "Shanuel's Lemma." Subsequently it was discovered that H. Fitting had proven Shanuel's Lemma in 1936 [F36] (with "projective" replaced by "free") as part of his study of the Fitting Invariants of a module.

$\mathrm{Tor}_*(k, k)$ *for local rings.*

Consider a local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Cartan and Eilenberg had shown that $\mathrm{Tor}_*^R(k, k)$ was a graded-commutative k-algebra [CE, XI.4–5]. Its Hilbert function is just the sequence of Betti numbers $b_i = \dim \mathrm{Tor}_i^R(k, k)$, and it is natural to consider the *Poincaré-Betti series* $P_R(t) = \sum_{i=0}^{\infty} b_i t^i$. Note that the first Betti number is $b_1 = \dim(\mathfrak{m}/\mathfrak{m}^2)$. For example, if R is a regular ring, it was well known that $\mathrm{Tor}_*^R(k, k)$ was an exterior algebra, so that $P_R(t) = (1 + t)^{b_1}$.

Serre showed in 1955 [Se56] that one always had $P_R(t) \geq (1 + t)^{b_1}$, i.e., that b_i is at least $\binom{b_1}{i}$. In particular, if $i = b_1$ then $b_i \geq 1$ and so $\mathrm{Tor}_{b_1}^R(k, k) \neq 0$. As we mentioned above, this was the key step in Serre's proof that local rings of finite global dimension are regular. In his 1956 study [T57], Tate showed that k had a free R-module resolution F_* which was a graded-commutative differential graded algebra, and used this to show that if R is not regular then $P_R(t) \geq (1+t)^{b_1}/(1-t^2)$, i.e., that b_i is at least $\binom{b_1}{i} + \binom{b_1}{i-2} + \cdots$. This is the best lower bound. In case R is the quotient of a regular local ring by a regular sequence of length r (contained in the square of the maximal ideal), Tate showed that the Poincaré-Betti series of R is the rational function $P_R(t) = (1 + t)^{b_1}/(1 - t^2)^r$.

Based upon Tate's results, Serre stated on p. 118 of [SeM] that it was not known whether or not $P_R(t)$ was always a rational function. This problem remained open for over twenty years, until it was settled negatively by David Anick [A82]. Anick's example was an artinian algebra R with $\mathfrak{m}^3 = 0$. Constructing a finite simply-connected CW complex X whose cohomology ring was R, a result of Roos [R79] showed that the Poincaré-Betti series of the loop space ΩX,

$H(t) = \sum \dim H_i(\Omega X)\, t^i$ was not a rational function either. This settled a second problem of Serre, also posed on p. 118 of [SeM].

Matlis Duality.

In his 1958 thesis [M58] under Kaplansky, Eben Matlis studied the structure of injective modules over a noetherian ring R, and showed that they can be written uniquely as direct sums of copies of the injective hulls $E(R/\mathfrak{p})$, as $\mathfrak{p}$ ranges over the prime ideals of R. This put injective resolutions on an equal footing with projective resolutions.

Let $\mathcal{A}$ denote an additive category of modules over a ring R. A *dualizing functor* on $\mathcal{A}$ is an exact contravariant R-linear functor D from $\mathcal{A}$ to itself such that $D(D(M)) = M$. Matlis' thesis [M58] also showed that the category $\mathcal{A}$ of modules of finite length over a local noetherian ring R has a unique dualizing functor: $D(M) = \mathrm{Hom}_R(M, E)$, where E is the injective hull of $R/\mathfrak{m}$.

This turned attention to other kinds of duality, and to modules of finite injective dimension. The goal here was to find the analogue of Serre's Duality Theorem for projective space $X = \mathbb{P}^d$ [Se55]: if F is a coherent sheaf on X then the dual of the vector space $H^i(X; F)$ is $\mathrm{Ext}_X^{d-i}(F, \omega_X)$, where $\omega_X = \Omega_X^d$ is the sheaf of differential d-forms on X.

It would turn out that that the good class of rings from this perspective would be Gorenstein rings. In a 1957 Séminaire Bourbaki talk on Duality ([GFGA, exp. 2], Grothendieck defined a commutative ring R (or scheme) of finite type over a field to be "Gorenstein" if it is Cohen-Macaulay and a certain R-module ω_R is locally free of rank 1. A few years later, Bass proved a theorem characterizing rings of finite self-injective dimension, and Serre remarked that the two definitions agreed in a geometric context. Bass then consolidated these notions in [B62], giving the modern definition: a commutative noetherian ring R is called *Gorenstein* if all its local rings have finite injective dimension. Bass proved that this is equivalent to several other conditions, such as R being Cohen-Macaulay and a system of parameters generates an irreducible ideal in each local ring. Nowadays we have the notion of the *canonical module* ω_R of a ring (see below), and if R is a Cohen-Macaulay local ring, then R is Gorenstein if and only if R is its own canonical module: $\omega_R = R$. For example, in Matlis Duality for a zero-dimensional ring, the role of ω_R is played by E, and R is Gorenstein exactly when $E = R$.

Local Cohomology and Duality.

In 1961, Grothendieck ran a Harvard seminar on Local Cohomology, based upon his 1957 Séminaire Bourbaki talk on Duality ([GFGA, exp. 2]); the notes were eventually published in [G67]. From the viewpoint of schemes, the local cohomology of a sheaf is the same as cohomology with supports. From the viewpoint of noetherian local rings, the local cohomology $H^*_{\mathfrak{m}}(M)$ of a module M are the derived functors of the $\mathfrak{m}$-primary submodule functor $H^0_{\mathfrak{m}}(M) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{m}^n, M)$, so $H^i_{\mathfrak{m}}(M) = \varinjlim \mathrm{Ext}^i_R(R/\mathfrak{m}^n, M)$.

Grothendieck showed that the depth of M is characterized as the smallest i such that $H^i_{\mathfrak{m}}(M) \neq 0$, and that if R is a Cohen-Macaulay ring then $H^i_{\mathfrak{m}}(R) \neq 0$ only for $i = \dim(R)$. Moreover, R is a Gorenstein ring if and only if the module $H_{\mathfrak{m}}^{\dim(R)}(R)$ is *dualizing* in Matlis' sense, meaning that it is the injective hull of $R/\mathfrak{m}$.

The highlight of the seminar was the Duality Theorem: if R is a complete Gorenstein ring of dimension d, then $H^i_{\mathfrak{m}}(M)$ is dual to $\mathrm{Ext}^{d-i}_R(M,R)$, in the sense that Matlis' dualizing functor D interchanges them. For a more general local ring, the duality is more complicated. If R is complete and Cohen-Macauley, one considers the functors $T^i(M) = D(H^i_{\mathfrak{m}}(M))$, and shows that they equal $\mathrm{Ext}^{d-i}_R(M,\omega_R)$, where $\omega_R = D(H^d_{\mathfrak{m}}(R))$. More generally, Grothendieck also observed that the $T^i(M)$ may be interpreted as $\mathrm{Ext}^{d-i}_R(M,K_R)$ for a suitable dualizing cochain complex K_R on R [G67, 6.8]. This led to the development of the derived category $D(R)$, which we shall describe shortly.

This material on Duality took awhile to absorb, and a ring-theoretic derivation of these results only appeared in 1970 [S70]. Gradually the notion of a *canonical module* ω_R became the organizing principal for duality theory, and R is Gorenstein exactly when $\omega_R = R$. If R is Cohen-Macaulay, the canonical module is defined [HK71] to be a maximal Cohen-Macaulay R-module of finite injective dimension, and the functor $D(M) = \mathrm{Hom}_R(M,\omega_R)$ is dualizing on the category of maximal Cohen-Macaulay R-modules.

In 1971, Sharp [S71] used local cohomology (and duality) to show that if R is a complete Cohen-Macauley local ring then the Gorenstein modules are precisely the direct sums of ω_R. He also showed that the final term in the Cousin complex of an R-module M is $H^{\dim(M)}_{\mathfrak{m}}(M)$.

In 1976 Hochster and Roberts [HR76] studied the local cohomology of a graded ring R in characteristic $p > 0$, and found that the structure of the local cohomology $H^i_{\mathfrak{m}}(R)$ was amazingly simplified under certain assumptions, such as the purity of the Frobenius homomorphism $F\colon R \to R$. They were also able to lift these characteristic p results to certain rings of characteristic 0, beginning a rennaisance in the study of Cohen-Macaulay rings.

Cohomology Theories in Algebraic Geometry.

During the early 1950's, the foundations of algebraic geometry were reworked by O. Zariski and others, focussing upon the role played by the algebras of regular functions. In his classic paper [Se55], Serre observed that if U is affine, with coordinate ring R, then there is an equivalence between finitely generated R-modules and coherent sheaves of modules on U. Hence restriction to an affine open V of U is an exact functor on coherent modules, because it corresponds to localization of modules. This implies that if F is coherent and U is affine then the Čech cohomology $\check{H}^q(U,F)$ vanishes. Using this, Serre defined the cohomology groups $H^q(X,F)$ of a coherent module on any variety X as the Čech cohomology relative to a covering of X by affine open subvarieties U. All this was in the spirit of the Cartan Seminars on Sheaf Theory in 1948–1950, but with the homological underpinnings of Cartan-Eilenberg available, Serre's presentation in terms of the Zariski topology was much simpler.

Serre also proved in [GAGA] that if X is a projective variety over $\mathbb{C}$ the groups $H^q(X,F)$ were the same as the analytically defined Betti cohomology, leaving little doubt that using the Zariski topology was a good approach to cohomology.

Grothendieck then observed that Serre's construction was a special case of the derived functor sheaf cohomology (for the Zariski topology) that he had developed in [G57]. Chapter III of [EGA] was devoted to the Zariski cohomology theory of coherent sheaves on a scheme, using the right derived functors Rf_* associated to a morphism $f\colon X \to Y$.

As part of the preliminaries to this development, Grothendieck wrote a primer on spectral sequences and hypercohomology in [EGA, 0_{III}]. This was a reworking of the corresponding material in [CE] and [G57] into a more workable form, and made these tools widely available to algebraic geometers.

Galois cohomology.

We have already mentioned that Hochschild [Hh50] coined the term "Galois cohomology" for the group cohomology of the Galois groups $G = Gal(K/k)$, where K is a (possibly infinite) Galois field extension of k. As we have already mentioned, Hochschild and Tate ([T52], [T54]) applied Galois cohomology to class field theory.

In the 1950's Tate began to systematically study what he called the "Galois cohomology" of the Galois groups $G = Gal(K/k)$, where K is a (possibly infinite) Galois field extension of k, such as the separable closure of k. Such a group has a topology induced by its finite quotients: $G = \varprojlim G/H_F$, where F ranges over all the finite extensions of K contained in k and $H_F = Gal(K/F)$. As a topological group, G is compact, Hausdorff and totally disconnected; today we call such groups *profinite*. Moreover, each H_F is an open subgroup of finite index in G.

In 1954, Kawada and Tate [KT55] used Galois cohomology to calculate the cohomology of a variety. To an étale covering U of X they associated a subgroup of the Galois group of $k(U)/k(X)$. This would later be recognized as the first use of what would later be called étale cohomology.

After years of gestation, a published account of Galois cohomology appeared in the 1958 paper [LT] by Serge Lang and John Tate. One considers a G-module A which is *discrete* in the sense that the action $G \times A \to A$ is continuous (when A has the discrete topology), and defines the *Galois cohomology* $H^*(G, A)$ to be the cohomology of the complex $C^*(G, A)$ of continuous cochains, that is, maps $\phi\colon G^n \to A$ which are continuous. An almost immediate observation is that

$$H^*(G, A) = \varinjlim_{H} H^*(G/H, A^H)$$

as H ranges through the open subgroups of finite index in G.

Tate's applications lay in the cases where $\mathbf{A}$ is an abelian group scheme defined over k; the G-module in this case is $A = \mathbf{A}(\bar{k})$, the group of rational points over the separable closure $\bar{k}$ of k.

One of the most important examples is the group scheme $\mathbf{A} = \mathbf{G}_m$, for which the G-module A is $\bar{k}^\times = \mathbf{G}_m(\bar{k})$ of units of $\bar{k}$. Hilbert's "Theorem 90" states that for every finite Galois extension F/k we have $H^1(Gal(F/k), F^\times) = 0$; taking the direct limit over all such F and setting $G = Gal(\bar{k}/k)$ yields the infinite version $H^1(G, \bar{k}^\times) = 0$. As we have seen, it was already known that $H^2(Gal(F/k), F^\times)$ is the relative Brauer group $Br(F/k)$; taking the direct limit over all such F shows that $H^2(F, \mathbf{G}_m)$ is the classical Brauer group $Br(F)$ introduced by Richard Brauer [B28] and Brauer-Noether [BN].

Serre's 1962 course *Cohomologie galoisienne* [SeCG], published in 1964, has remained the standard reference on the Galois cohomology over number fields.

Étale cohomology.

In 1958, Grothendieck found a common generalization of Galois cohomology and Zariski cohomology and used it to define the étale cohomology of schemes. A *Grothendieck topology* is a category $\mathcal{T}$ such that each object X is equipped with a family of morphisms $\{U_i \to X\}$, called *coverings*, subject to certain axioms. From this viewpoint, a *sheaf* F is a contravariant functor on $\mathcal{T}$ such that for each covering, each $s \in F(X)$ is uniquely determined by elements $s_i \in F(U_i)$ which agree in each $F(U_i \times_X U_j)$. The category of sheaves of abelian groups on $\mathcal{T}$ is an abelian category with enough injectives, and Grothendieck defined the cohomology groups $H^*(\mathcal{T}, F)$ to be the right derived functors of $F \mapsto F(X)$. When X is a topological space and $\mathcal{T}$ is the poset of open subspaces then sheaf has its usual meaning, and we recover the usual sheaf cohomology on X.

To define the *étale topology* on a scheme X, Grothendieck took the category of all schemes U which are étale over X, with the set-theoretic notion of covering. If F is a sheaf for this topology, the above construction defines the étale cohomology groups $H^*(X_{et}, F)$ of F on X. When X is the spectrum of a field k and $G = Gal(\bar{k}/k)$, a discrete G-module A is the same as an étale sheaf on X, so the étale cohomology of X with coefficients A agrees with Tate's Galois cohomology $H^*(G, A)$.

In Fall 1961, Grothendieck presented his ideas in a course at Harvard. The following semester (Spring 1962), M. Artin ran a seminar covering Grothendieck Topologies, as well as some material on étale cohomology (such as cohomological dimension). The published notes [A62] of this seminar, as well as Giraud's Bourbaki talk [Gir63] made the ideas available to a wide audience.

The next year (1962–63), when the seminar continued in France, Artin and Grothendieck worked out the fundamental structure theorems of étale cohomology: proper and smooth base change, specialization, cohomology with compact supports and duality. The following year, more results were obtained (such as purity and the Lefschetz trace formula), with the seminar notes eventually appearing as [SGA4].

One of Grothendieck's early successes with étale cohomology was his cohomological proof of the rationality of the Zeta function $Z_X(t)$ of a scheme of finite type over the finite field $\mathbb{F}_q$. He proved that each factor $P_i(t)$ of $Z_X(t)$ is the characteristic polynomial of the Frobenius operator acting on an l-adic cohomology group, namely $H^i(X, \mathbb{Q}_l) = \varinjlim H^i_{et}(X, \mathbb{Z}/(l^\nu))$. In 1972, Deligne used étale cohomology to prove the "Riemann hypothesis" over $\mathbb{F}_q$ [D74]: the eigenvalues of the Frobenius on $H^i(X, \mathbb{Q}_l)$ (and hence the zeroes and poles of the zeta function) were algebraic integers with absolute value $q^{i/2}$. This completed the proof of the celebrated Weil Conjectures, and firmly established the importance of étale cohomology.

Derived Categories.

After Grothendieck's 1961 Harvard seminar on Local Cohomology, described above, Grothendieck realized that in order to extend these results to arbitrary schemes he needed some results in homological algebra which were not yet available. This was overcome by Verdier's 1963 thesis [V] on Derived Categories.

The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category obtained from the category $\mathbf{Ch}(\mathcal{A})$ of (co)chain complexes by formally inverting the quasi-isomorphisms, i.e., the maps $C \to C'$ which induce isomorphisms on (co)homology.

To describe it, Verdier introduced the notion of a *triangulated category*. The quotient category $K(\mathcal{A})$ of $\mathbf{Ch}(\mathcal{A})$, whose morphisms are the chain homotopy equivalence classes of maps, is triangulated; $D(\mathcal{A})$, which is formed from $K(\mathcal{A})$ by a calculus of fractions, is also triangulated. If $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor then under reasonable conditions there is a functor $\mathbf{R}F\colon D(\mathcal{A}) \to D(\mathcal{B})$ with the property that if an A in $\mathcal{A}$ is considered as a complex then the cohomology of the complex $\mathbf{R}F(A)$ give the ordinary right derived functors $R^*F(A)$.

The topologist D. Puppe had already defined the notion of a *stable category* in [P62]. This is just a graded triangulated category without the "octahedral" axiom. Since Puppe only discussed $K(\mathcal{A})$ and not $D(\mathcal{A})$, and did not deal with the total derived functors $\mathbf{R}F$, his notion never caught the attention of the algebraists.

In the Summer of 1963, after Hartshorne proposed to run a seminar at Harvard on duality theory, Grothendieck wrote a series of "prenotes," sketching the construction of a functor $f^!\colon D(Y\text{-}\mathbf{mod}) \to D(X\text{-}\mathbf{mod})$ associated to a reasonable morphism $f\colon X \to Y$ of schemes, together with a natural trace morphism $\mathbf{R}f_*f^!(A) \to A$. The so-called "Séminaire Hartshorne" was held at Harvard in 1963–64, based upon these prenotes, and the seminar notes appeared as [H66]. An appendix to [H66], written by Deligne in 1966, constructs $f^!$ for every separated morphism of finite type between noetherian schemes.

During the 1966–67 Séminaire de Geometrie Algebrique [SGA6], Grothendieck used the triangulated category $\mathrm{Perf}(X)$ of *perfect* complexes of $\mathcal{O}_X$-modules to develop a global theory of intersections and a Riemann-Roch Theorem for arbitrary noetherian schemes. By definition, a complex is perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles, and the alternating sum of these vector bundles gives a well-defined element in the Grothendieck group $K(X)$, at least if X is quasi-projective or smooth. If $f\colon X \to Y$ is proper, the machinery of triangulated categories yields an exact functor $Rf_*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$ and hence a homomorphism $K(X) \to K(Y)$.

In 1978, Bernstein-Gelfand-Gelfand [BBG] used derived categories to classify vector bundles on projective space $\mathbb{P}^n$ over a field k in terms of graded modules over the exterior algebra Λ on $n+1$ variables. The crucial step in their classification was the discovery of an isomorphism between the (bounded) derived categories of graded modules $D^b_{gr}(\Lambda)$ and $D^b_{gr}(R)$, where R is the polynomial algebra on $n+1$ variables. This result showed that $D^b(\mathcal{A})$ did not determine the "heart" category $\mathcal{A}$, a result which came as a bit of a surprise.

The problem of multiple hearts for a triangulated category was revisited in 1982 by Bernstein-Beilinson-Deligne [BBD]. These authors used triangulated categories to study $\mathcal{D}$-modules and perverse sheaves on a stratified space. In 1988, Beilinson-Ginsburg-Schechtman [BGS] generalized the results of [BBG] and [BBD] by proving that many filtered triangulated categories have two hearts, which are in Koszul duality.

In the mid-1980's, derived categories found yet another application. The notion of a tilting module had come up in the study of representations of finite algebras. Cline-Parshall-Scott [CPS] showed that if T is a tilting module for A, and $B = \mathrm{Hom}_A(T,T)$, then $D^b(A) \cong D^b(B)$.

Early work on derived categories was often restricted to either bounded or bounded below complexes, because of the need to work with injective (or projective) resolutions. In 1988, Spaltenstein [S88] showed that every unbounded complex was quasi-isomorphic to a “fibrant” complex, and that one could use fibrant complexes to compute derived functors. This result has led to several new developments which continue to this day.

Simplicial Methods

During the 1940's, Eilenberg kept encountering things called “abstract complexes” which resembled the triangulated polyhedra (or “geometric simplicial complexes”) introduced by Poincaré, except that a simplex was not always determined by its faces. For example the abstract complex $K(\pi)$ of [EM43] and the singular complex $S(X)$ of [E44] had this property. To describe this phenomenon, Eilenber and Zilber [EZ50] introduced the notions of a *semi-simplicial* complex and a *complete semi-simplicial* complex in 1950. The Eilenberg-Zilber notion of a complete semi-simplicial complex is identical to our modern notion of a *simplicial set* K: it is a sequence $K_0, K_1, \dots$ of sets together with face maps $\partial_i \colon K_q \to K_{q-1}$ and degeneracy maps $s_i \colon K_q \to K_{q+1}$ $(0 \le i \le q)$ satisfying certain axioms; a semi-simplicial complex is just a simplicial set without the degeneracy maps.

A word about changing terminology is in order. The term “complete semi-simplicial complex” was awkward and was quickly abbreviated to “c.s.s. complex.” During the 1950's the term c.s.s. complex prevailed, although the short-lived term “FD-complex” was also used in [EM54] and [D58]. Largely due to the influence of John Moore, the adjective “complete” began to be omitted, starting with 1954, while the notion of “semi-simplicial complex” languished in obscurity. By the early 1960's the term “semi-simplicial set” had replaced “c.s.s. complex.” By the late 1960's, even the prefix “semi” was dropped, influenced by the book [May]; since then “simplicial set” has been the universally used term.

Returning to the early 1950's, we mention two results which showed the power of the new simplicial methods. The “Eilenberg-Zilber Theorem” was proven in 1953 [EZ53] as an application of c.s.s. complexes to products: the (simplicial) map $S(X \times Y) \simeq S(X) \otimes S(Y)$, implicitly defined by Alexander and Whitney in 1935, is a homotopy equivalence. In 1955, the homotopy theory of c.s.s. complexes satisfying an extension condition was developed by Daniel Kan [Kan56]; a simplicial set satisfying Kan's extension condition is now called a *Kan complex*.

Homotopical algebra.

The homological study of simplicial abelian groups was launched by Eilenberg and Mac Lane in [EM54], as part of their algebraic program to find the cohomology of Eilenberg-Mac Lane spaces $K(\pi, n)$. This program was analyzed with typical thoroughness in the 1954/55 Seminaire Cartan [C55]. In exposés 18 and 19 of that seminar, John Moore showed that every simplicial group K is a Kan complex, and that one could compute its homotopy groups as the homology of a chain complex N_* of groups, where $N_q \subset K_q$ is the intersection of kernels of all the face maps except ∂_q. The complex N_* quickly became known as the *Moore complex* of K.

In 1956–57, A. Dold [D58] and D. Kan [Kan58] independently discovered that the Moore complex provided an equivalence between the category of simplicial abelian groups and the category of non-negative chain complexes of abelian groups. This

Dold-Kan correspondence was later codified in [DP]. Under the correspondence, Moore's result states that simplicial homotopy corresponds to homology. With this correspondence at hand, simplicial techniques could be brought to bear on any homological problem.

Dold and Puppe [DP] announced in 1958 that with simplicial methods one could define the derived functors of a non-additive functor T (say of modules); their detailed paper appeared in 1961. The key idea was that one could consider a projective resolution P_* of a module M as a simplicial module via the Dold-Kan correspondence. Since the notion of simplicial homotopy doesn't involve addition, we may take the homotopy groups of $T(P_*)$ as the derived functors $L_iT(M)$ of T. A variant is obtained by placing M in degree $n > 0$; the derived functors $L_iT(M, n)$ are the homotopy groups of $T(P[n])$, where the simplicial module $P[n]$ corresponds to the chain complex P_* shifted n places. For example, the ith homology $H_i(K(\pi, n); \mathbb{Z})$ of an Eilenberg-Mac Lane space $K(\pi, n)$ is just $L_iT(\pi, n)$ for the group ring functor $T(\pi) = \mathbb{Z}[\pi]$.

It is possible to generalize the Dold-Puppe construction and define the left derived functors of any functor T from any category $\mathcal{C}$ to an abelian category, as long as $\mathcal{C}$ is closed under finite limits and has enough projective objects. This observation evolved during the late 1960's, finding voice in M. André's book [A67], Quillen's book [Q67] on homotopical algebra, and in the later papers [A70, Q70]. In fact there are three standard constructions, which agree in reasonable situations.

André's construction [A67] uses a subcategory of "acyclic models" in $\mathcal{C}$. In the category of functors on $\mathcal{C}$, one finds a resolution $T_* \to T$ which is aspherical on the "model" objects. Then one defines $L_iT(A)$ to be $\pi_iT_*(A)$, or H_i of the chain complex associated to the simplicial module $T_*(A)$.

Quillen's construction is simpler: one finds a simplicial "resolution" $P_* \to A$ of each A in $\mathcal{C}$, and defines $L_iT(A)$ to be $H_iT(P_*)$. The work comes in deciding what a "resolution" is: P_* should be *cofibrant* and $P_* \to A$ should be an *acyclic fibration* in the terminology of [Q67]. In many algebriac applications, fibrations are defined by a relative lifting property, so all "relatively projective" objects are cofibrant.

During 1965–69, Barr and Beck [BB] developed the idea of cotriple resolutions as a functorial way to obtain resolutions for computing nonabelian derived functors. Suppose that there is a forgetful functor $U \colon \mathcal{C} \to \mathcal{S}$ with a left adjoint F. Then the functor FU is called a *cotriple*, and the iterates $P_i = (FU)^{i+1}(A)$ often form a simplicial "resolution" $P_* \to A$. Again, one takes $L_iT(A) = H_iT(P_*)$.

Cohomology of commutative rings.

In analogy with Hochschild's (co)homology theory for associative algebras, it is reasonable to ask for a (co)homology theory for commutative rings. Let $k \to A$ be a map of commutative rings, and M an A-module. Then Hochschild's group $H^1(A; M)$ is the A-module $Der_k(A, M)$ of all derivations $A \to M$ which vanish on k (as there are no inner derivations), $H_1(A; M)$ is $M \otimes \Omega_{A/k}$ and $H^2(A; M)$ classifies all associative k-algebra extensions B of A by M which are *k-split*, meaning that $B \cong A \oplus M$ as a k-module (this condition is obvious when k is a field). What was wanted was a theory with the same H^1 and H_1, but such that H^2 was the group $\mathrm{Exalcomm}_k(A, M)$ classifying all commutative k-algebra extensions of A by M.

The functors H^1 and H^2 were first studied by P. Cartier [C56] in the case that

$A = K$ is a field extension of k, and partially extended to commutative rings by Nakai [N61]. In a 1961 course at Harvard, Grothendieck defined $\mathrm{Exalcomm}_k(A, M)$ and constructed a 6-term cohomology sequence for $k \to A \to B$ [EGA $0_{IV}(18.4.2)$].

When k is a field, Harrison [H62] used a subcomplex of the Hochschild complex to define k-modules $H^*_{harr}(A, M)$ with $H^1_{harr} = H^1$ and $H^2_{harr} = \mathrm{Exalcomm}_k$, equipped with a 9-term cohomology sequence. When k is perfect, and A is the local ring (at some point) of a variety over k, Harrison proved the following two results: (1) A is regular if and only if $H^2_{harr}(A, -) = 0$, and (2) A is a complete intersection if and only if $\dim H^1_{harr}(A, A/\mathfrak{m}) - \dim H^2_{harr}(A, A/\mathfrak{m}) = \dim A$.

The next step was taken in the 1964 paper [LS] by two Ph. D. students of Tate, Lichtenbaum and Schlessinger. Let k be any commutative ring. For each commutative ring map $f\colon k \to A$, they defined a 3-term chain complex $\mathbf{L}^{\cdot}$, called the *cotangent complex* of f, and — for $i = 0, 1, 2$ — set $T_i(A/k, M) = H_i(\mathbf{L}^{\cdot} \otimes M)$, $T^i(A/k, M) = H^i \operatorname{Hom}(\mathbf{L}^{\cdot}, M)$. When k is a field the $T^i(A/k, M)$ agreed with Harrison's $H^{i+1}_{harr}(A, M)$, and in general $T^1(A/k, M) = \mathrm{Exalcomm}_k(A, M)$. Their infinitesimal criterion for A/k to be smooth, in terms of the vanishing of $T^1(A/k)$, was later used by Grothendieck to great advantage in [EGA IV.17]. If k is noetherian, R is a localization of $k[x, \dots, y]$ and $A = R/I$, they showed that $T^2(A/k, -) = 0$ if and only if A is a complete intersection, i.e., I is defined by a regular sequence in R. Schlessinger's thesis applied the T^i to deformation theory, while Lichtenbaum's thesis was concerned with applications to relative intersection theory.

In 1967, M. André [A67, A70, A74] and Quillen [Q70] discovered what we now call André-Quillen cohomology. If $k \to A$ and M are as above, their groups $D^i(A/k, M)$ agree with the Lichtenbaum-Schlessinger groups $T^i(A/k, M)$ for $i = 0, 1, 2$. It comes with a long exact sequence for $k \to A \to B$ (generalizing Harrison's) and generalizations of the Lichtenbaum-Schlessinger results for smoothness and local complete intersections. In this theory, the central role is played by a simplicial A-module $\mathbb{L}_{A/k}$, called the *cotangent complex* of A relative to k, because of the similarity (using the Dold-Kan correspondence) to the Lichtenbaum-Schlessinger complex $\mathbf{L}^{\cdot}$. This complex is well-defined in the derived category of chain complexes of A-modules, and one has $D^i(A/k, M) = H^i \operatorname{Hom}_A(\mathbb{L}_{A/k}, M)$ and $D_i(A/k, M) = H_i(\mathbb{L}_{A/k} \otimes_A M)$.

Formally, the $D^i(A/k, M)$ are the nonabelian derived functors of the functor $T(B) = Der_k(B, M) \cong \operatorname{Hom}_A(A \otimes_B \Omega_{B/k})$ on the category $\mathcal{C}$ of commutative k-algebras over A. According to the above prescription, the definition starts with an acyclic simplicial resolution $P_* \to A$ in $\mathcal{C}$, and has $D^i(A/k, M) = H^i Der_k(P_*, M)$. Defining the simplicial A-module $\mathbb{L}_{A/k} = A \otimes_P \Omega_{P/k}$, a little algebra yields the above formulas.

Higher algebraic K-theory.

In order to find a possible definition of the higher K-groups $K_n(R)$ of a ring R, Swan was led in 1968 to consider the nonabelian derived functors of the general linear group GL on the category of rings [Sw70]. This required a slight generalization of derived functor, since the category of groups is not an abelian category. In this context we have a functor G from a category $\mathcal{C}$, such as the category of rings, to the category of groups or sets.

Swan's original construction followed André's method, finding an acyclic resolution $G_* \to GL$ in the functor category and setting $K_n(R) = \pi_{n-2}G_*(R)$ for $n \geq 2$. In 1969 Gersten gave a cotriple construction [G71], using the cotriple associated to the forgetful functor from rings to sets, while both Keune [K71] and Swan [Sw72] gave constructions using free resolutions $P_* \to R$ to define $K_n(R) = \pi_{n-2}GL(P_*)$ for $n \geq 2$. By 1970, Swan had proven [Sw72] that all three constructions yielded the same functors $K_n(R)$.

Historically, however, the important construction was given by Quillen in 1969 [Q71]. He showed how to modify the classifying space $BGL(R)$ of $GL(R)$ to obtain a topological space $BGL(R)^+$ with the same homology as $BGL(R)$, and defined $K_n(R) = \pi_n BGL(R)^+$ for $n \geq 1$. The equivalence of Quillen's topological definition with the homological Swan-Gersten definition was established in 1972 by combining partial results obtained by several authors [A73]. Since then the field of higher algebraic K-theory has taken on a life of its own, but that is another story.

Hochschild and cyclic homology.

We have already described the 1945 development [Hh45] of Hochschild homology of an algebra A over a field k. The next step was to let A be an algebra over an arbitrary commutative base ring k. In his 1956 paper [Hh56], Hochschild began a systematic study of exact sequences of R-modules which are k-split (split as sequences of k-modules). This became part of a "relative" homological algebra movement.

Hochschild, Kostant and Rosenberg showed in 1962 [HKR] that if A is smooth of finite type over a field k, then there is a natural isomorphism $\Omega^*_{A/k} \cong H_*(A, A)$. It follows that for such A there is an analogue $d\colon \Omega^n_A \to \Omega^{n+1}_A$ of de Rham's operator for manifolds. Rinehart [R63] mimicked this construction for all algebras, constructing a chain map B inducing an operator $HH_n(A, A) \to H_{n+1}(A, A)$. This attempt to define an analogue of de Rham cohomology was before its time: twenty years later Alain Connes [C85] as well as Feigin and Tsygan [T83, FT] would both seize upon B and make it the foundation of cyclic homology, unaware of Rinehart's earlier work.

We end our quick tour by mentioning an important application, discovered by Gerstenhaber in the 1964 paper [G64]. A *deformation* of an associative algebra A is a $k[[t]]$-algebra structure on the $k[[t]]$-module $A[[t]]$ whose product agrees modulo t with the given product on A. Reducing a deformation modulo t^2 yields a k-split algebra extension of A by A, so giving the "infinitesimal" part of the deformation is equivalent to giving an element of $H^2(A, A)$. Gerstenhaber showed that there is a whole sequence of obstructions to deformations of A, lying in the Hochschild cohomology group $H^3(A, A)$. If A is smooth of finite type, the Hochschild-Kostant-Rosenberg theorem implies that the obstructions belong to $\Omega^3_{A/k}$.

Cotor *for coalgebras.*

Hochschild homology was also involved in the early development of (differential graded) coalgebras over a field. This field was heavily influenced by its applications to topology, in part because the homology of a topological space X is a graded coalgebra, via the diagonal map $H_*(X) \to H_*(X \times X) \cong H_*(X) \otimes H_*(X)$. Moreover, the normalized chain complex $C_*(X)$ is a differential graded coalgebra.

In 1956, J. F. Adams [A56] discovered a recipe for the homology of the loop space ΩX when X is simply connected. To describe it, he considered $C_*(X)$ as a differential graded coalgebra. Mimicking the Eilenberg-Mac Lane bar construction, Adams defined a differential graded algebra F_*, called the *cobar construction*, and showed that $H_*(\Omega X) \cong H_*(F_*)$. This purely algebraic construction attracted the attention of topologists to the algebraic structure of coalgebras and their comodules.

Now if C is a coalgebra one can define the *cotensor product* $M \Box_C N$ of comodules M and N. Its right derived functors are called the *cotorsion* products $\mathrm{Cotor}^C(M,N)$ of M and N. In [EM66], Eilenberg and Moore defined and studied the cotensor product over a DG coalgebra $C = C_*$. Under mild flatness hypotheses, they constructed what we now call the "Eilenberg-Moore spectral sequence," which has E^2 equal to $\mathrm{Cotor}_{pq}^{HC}(H(M), H(N))$ and converges to $\mathrm{Cotor}^C(M,N)$. The importance of this is illustrated by the case when C is the normalized chain complex of a simply connected topological space X, and M and N are the chain complexes of spaces E and X' over X. If $E \to X$ is a Serre fibration, they prove that $\mathrm{Cotor}^C(M,N)$ is the homology of the fiber space $E' = E \times_X X'$, so this provides a powerful method to calculate homology. Of course when X' and E are contractible then $E' \simeq \Omega X$, and they recover Adams' cobar construction.

Eilenberg and Moore also studied the dual construction for tensor products of differential graded modules M, N over a differential graded algebra R. In this case the spectral sequence is $E_2^{pq} = \mathrm{Tor}_{H(R)}(H(N), H(M)) \Rightarrow \mathrm{Tor}_R(N,M)$. Using the cochain algebras in the above topological situation, Eilenberg and Moore proved that $H^*(E') \cong \mathrm{Tor}_{C^*(X)}(C^*(E), C^*(X'))$, so the spectral sequence converges to $H^*(E')$. This spectral sequence was described and studied in [S67] by Larry Smith, who showed that this spectral sequence often collapsed.

Here is one application. Suppose that Y is simply connected and we take $X = Y \times Y$, with X' the diagonal copy of Y, and E the path space of Y. Then $E' = \Omega Y$ and if $C^*(Y)$ takes coefficients in a field k the Künneth formula yields $C^*(X) \simeq C^*(Y) \otimes C^*(Y)$. Since the Eilenberg-Moore spectral sequence collapses in this case it yields an isomorphism between $H^*(\Omega Y)$, and the Hochschild cohomology $HH^*(C^*(Y), k)$ of the differential graded algebra $C^*(Y)$.

Mac Lane Cohomology and Topological Hochschild Homology.

Let A be an associative ring and M an A-bimodule. As we have mentioned above, the Hochschild cohomology group $H^2(A,M)$ only measures ring extensions of A by M whose underlying abelian group is $A \oplus M$. (One takes k to be $\mathbb{Z}$.) In order to measure *all* ring extensions of A by M, Mac Lane introduced what we now call *Mac Lane cohomology* in the 1956 paper [M56]. One may naturally define a differential graded ring $Q = Q_*(A)$ and an augmentation $Q \to A$. By definition, $HML_*(A,M)$ and $HML^*(A,M)$ are the Hochschild homology $H_*(Q,M)$ and cohomology $H^*(Q,M)$. As required, ring extensions correspond to elements of the group $HML^2(A,M)$.

A variant for k-algebras and their extensions was invented in 1961 by U. Shukla [Shuk], and is called *Shukla homology.* Shukla proved two comparison results: when k is a field, Shukla homology recovers Hochschild homology; when $k = \mathbb{Z}$, Shukla homology agrees with a homology theory defined by Mac Lane in 1958 (which is not Mac Lane homology, as asserted by Shukla).

Both Mac Lane cohomology and Shukla homology were almost completely forgotten for thirty years, except for some calculations by Breen in [B78]. In 1991, an innocuous paper by Jibladze and Pirashvili [JP91] proved that the Mac Lane homology of a ring A (and a module M) is $\mathrm{Tor}_*^{\mathcal{F}}(A\otimes, M\otimes)$ in the functor category $\mathcal{F} = \mathcal{F}(A)$ of functors from the category of fin. gen. free A-modules to the category of A-modules. Similarly, the Mac Lane cohomology of A is $\mathrm{Ext}_{\mathcal{F}}(A\otimes, M\otimes)$. This was to lead to an unexpected connection to algebraic K-theory and manifolds.

In the late 1970's, F. Waldhausen introduced a variant of algebraic K-theory, which he called *stable K-theory* [W78]. His construction was designed to study the homotopy theory of the diffeomorphism group of a manifold, and could be applied to a ring spectrum A as well as ordinary rings. Following this lead in the early 1980's, M. Bökstedt [Bö] introduced a variant $THH_*(A,A)$ of Hochschild homology for ring spectra, called *Topological Hochschild Homology.* It is roughly obtained by replacing rings by ring spectra and tensor products over k by smash products. In 1987, Waldhausen announced that stable K-theory of A was isomorphic to $THH(A)$, but the proof [DM94] took several years to appear.

Then in 1992, Pirashvili and Waldhausen [PW92] used the functor category interpretation to prove that the Mac Lane homology group $HML(A,A)$ was the same as $THH(A)$. This showed that homological algebra could be applied to calculate the topological invariants of Waldhausen and Bökstedt. A new and active field of research has been born out of this discovery.

Cyclic homology.

Cyclic homology arose simultaneously in several applications in the early 1980's.

While studying applications of C^*-algebras to differential geometry in 1981, Alain Connes was led to study Hochschild cochains which were invariant under cyclic permutations of its arguments [C83, C85]. Realizing that such "cyclic" cochains were preserved by the Hochschild coboundary gave him a new cohomology theory, rapidly christened $HC^*(A)$ and called the *cyclic cohomology* of A. Meanwhile, Boris Tsygan [T83] was studying the homology of the Lie algebra $\mathfrak{gl}(A)$ over a field k of characteristic zero, and discovered that the Hopf algebra $H_*(\mathfrak{gl}(A);k)$ was the tensor algebra on the homology groups $K_i^+(A)$ of the complex of all Hochschild chains invariant under cyclic permutation; the proof, and the cohomology version, appeared in the paper [FT] by Feigin and Tsygan. This description of $H_*(\mathfrak{gl}(A);k)$ was discovered independently by Loday and Quillen [LQ], and their paper made the new subject of cyclic homology accessible to a large audience.

Both Connes and Tsygan discovered the following key structural sequence relating cyclic homology to Hochschild homology; Rinehart's operator [R63] is the composition BI.

$$\cdots HC_{n+1}(A) \xrightarrow{S} HC_{n-1}(A) \xrightarrow{B} H_n(A,A) \xrightarrow{I} HC_n(A) \cdots$$

Using this sequence, Connes and others rediscovered and clarified the connection with de Rham cohomology; for smooth algebras $HC_n(A)$ is a product of de Rham cohomology groups, together with $\Omega^n_{A/k}/d\Omega^{n-1}_{A/k}$.

In retrospect, cyclic homology had been hinted at in several places: pseudo-isotopy theory [DHS], the homology of S^1-spaces and in algebraic K-theory [L81].

Other applications soon arose. For example, Goodwillie showed in [G86] that the cyclic homology (over $\mathbb{Q}$) of a nilpotent ideal I is isomorphic to the algebraic K-theory of I. Because of its diverse applications to other areas of mathematics, cyclic homology became quickly established as a flourishing field in its own right.

It is impossible to give an accurate historical perspective on current developments. As tempting as it is, I shall refrain from doing so. Perhaps in fifty years the history of homological algebra will be unrecognizable to us today. Let us hope so!

References

[A15] J. W. Alexander, *A proof of the invariance of certain constants in Analysis Situs*, Trans. AMS **16** (1915), 148–154.

[A26] J. W. Alexander, *Combinatorial analysis situs*, Trans. AMS **28** (1926), 301–329.

[A55] M. Auslander, *On the dimension of modules and algebras. III. Global dimension*, Nagoya Math. J. **9** (1955), 67–77.

[A56] J. F. Adams, *On the cobar construction for coalgebras*, Proc. Nat. Acad. Sci. **42** (1956), 409–412.

[A62] M. Artin, *Grothendieck Topologies*, Mimeographed seminar notes, Harvard Univ., 1962.

[A67] M. André, *Méthode simplicial en algèbre homologique et algèbre commutative*, Lecture Notes in Math. vol. 32, Springer Verlag, 1967.

[A70] M. André, *Homology of simplicial objects*, Proc. Sympos. Pure Math., vol. XVII, 1970, pp. 15–36.

[A73] D. Anderson, *Relationship among K-theories*, Lecture Notes in Math. vol. 341, Springer Verlag, 1973, pp. 57–72.

[A74] M. André, *Homologie des algèbres commutatives*, Springer Verlag, 1974.

[A82] D. Anick, *A counterexample to a conjecture of Serre*, Annals of Math. **115** (1982), 1–33.

[AB56] M. Auslander and D. Buchsbaum, *Homological dimension in local rings*, Trans. AMS **85** (1957), 390–405; *Homological dimension in noetherian rings*, Proc. Nat. Acad. Sci. **42** (1956), 36–38.

[AB58] M. Auslander and D. Buchsbaum, *Codimension and multiplicity*, Annals of Math. **68** (1958), 625–657.

[AB59] M. Auslander and D. Buchsbaum, *Unique factorization in regular local rings*, Trans. AMS **45** (1959), 733–734.

[ANT] E. Artin, C. Nesbitt and R. Thrall, *Rings with Minimum Condition*, Univ. Michigan Press, 1944.

[B28] R. Brauer, *Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen. I.*, Math. Z. **28** (1928), 677–696.

[B34] R. Baer, *Erweiterungen von Gruppen und ihren Isomorphismen*, Math. Zeit. **38** (1934), 375–416.

[B35] R. Brauer, *Sur les invariants intégraux des variétés répresentatives des groupes de Lie simples clos*, C. R. Acad. Sci. (Paris) **201** (1935), 419–421.

[B40] R. Baer, *Abelian groups that are direct summands of every containing abelian group*, Bull AMS (1940), 800–806.

[B42] N. Bourbaki, *Éléments de Mathématique. Part I.*, Livre II, Algèbre ch. 2, Hermann, 1942.

[B55] D. Buchsbaum, *Exact categories and duality*, Trans. AMS **80** (1955), 1–34.

[B61] H. Bass, *Projective modules over algebras*, Annals of Math. **73** (1961), 532–542.

[B63] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeit. **82** (1963), 8–28.

[B78] L. Breen, *Extensions du groupe additif*, Publ. I.H.E.S. **48** (1978), 39–125.

[B81] J. M. Boardman, *Conditionally convergent spectral sequences*, preprint (1981).

[BB] M. Barr and J. Beck, *Homology and standard constructions*, Lecture Notes in Math. vol. 80, Springer Verlag, 1969.

[BBD] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982).

[BBG] I. Bernšteĭn, I. Gelfand and S. Gelfand, *Algebraic vector bundles on* $\mathbb{P}^n$ *and problems of linear algebra*, Funkt. Anal. i Prilozhen. **12** (1978), 66–67.

[Betti] E. Betti, *Sopra gli spazi di un numero qualunque di dimensioni*, Ann. Mat. pura appl. **2/4** (1871), 140–158.

[BGS] A. Beĭlinson, V. Ginsburg and V. Schechtman, *Koszul duality*, J. Geom. Phys. **5** (1988), 317–350.

[BN] R. Brauer and E. Noether, *Über minimale Zerfällungskörper irreduzibler Darstellungen*, Sitzungsberichte Preußischen Akad. Wissen. (1927), 221–228.

[Boll] M. Bollinger, *Geschichtliche Entwicklung des Homologiebegriffs*, Archive for History of Exact Sciences **9** (1972), 94–170.

[Bö] M. Bökstedt, *Topological Hochschild homology*, preprint 1985 (unpublished).

[BS] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136.

[C28] E. Cartan, *Sur les nombres de Betti des espaces de groupes clos*, C. R. Acad. Sci. (Paris) **187** (1928), 196–198.

[C29] E. Cartan, *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologique de ces espaces*, Ann. Soc. Polon. Math. **8** (1929), 181–225.

[C32] E. Čech, *Théorie générale de l'homologie dans un espace quelconque*, Fundamenta Math. **19** (1932), 149–183.

[C35] ———, *Les groupes de Betti d'un complex infini*, Fundamenta Math. **25** (1935), 33–44.

[C48] H. Cartan, *Séminaire Henri Cartan de topologie algébrique, 1948/1949.*, Secrétariat mathématique, Paris, 1955.

[C50] H. Cartan, *Séminaire Henri Cartan de topologie algébrique, 1950/1951.*, Cohomologie des groupes, suite spectrale, faisceaux, Secrétariat mathématique, Paris, 1955.

[C55] H. Cartan, *Séminaire Henri Cartan, 1954/1955.*, Algèbre de Eilenberg-Mac Lane et homotopie, Secrétariat mathématique, Paris, 1955.

[C56] P. Cartier, *Dérivations dans les corps*, Exposé 13, Seminaire Cartan et Chevalley 1955/56, Secrétariat mathématique, Paris, 1956.

[C83] A. Connes, *Cohomologie cyclique et foncteurs* Ext^n, C. R. Acad. Sci. Paris **296** (1983), 953–958.

[C85] A. Connes, *Non-commutative differential geometry*, Publ. I.H.E.S. **62** (1985), 257–360.

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton U. Press, 1956.

[ChE] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. AMS **63** (1948), 85–124.

[CPS] E. Cline, B. Parshall, and L. Scott, *Derived categories and Morita theory*, J. Algebra **104** (1986), 397–409.

[D58] A. Dold, *Homology of symmetric products and other functors of complexes*, Annals of Math. **68** (1958), 54–80.

[D74] P. Deligne, *La conjecture de Weil. I*, Publ. Math. I.H.E.S. **43** (1974), 273–307.

[DHS] W. Dwyer, W.-C. Hsiang and R. Staffeldt, *Pseudo-isotopy and invariant theory*, Topology **19** (1980), 367–385.

[Dieu] J. Dieudonné, *A History of Algebraic and Differential Topology 1900–1960*, Birkhäuser, 1989.

[DM94] B. Dundas and R. McCarthy, *Stable K-theory and topological Hochschild homology*, Annals of Math. **140** (1994), 685–701.

[DP] A. Dold and D. Puppe, *Homologie nicht-additiver Funktoren; Anwendungen*, Ann. Inst. Fourier (Grenoble) **11** (1961), 201–312; *Non-additive functors, their derived functors, and the suspension homomorphism*, Proc. Nat. Acad. Sci. **44** (1958), 1065–1068.

[dR] G. de Rham, *Sur l'analysis situs des variétés à* n *dimensions*, J. Math. Pures Appl. **10** (1931), 115–200; C. R. Acad. Sci. (Paris) **188** (1929), 1651–1652.

[E44] S. Eilenberg, *Singular homology theory*, Annals of Math. **45** (1944), 407–447.

[E49] S. Eilenberg, *Topological methods in abstract algebra*, Bull. AMS **55** (1949).

[Eck] B. Eckmann, *Der Cohomologie-Ring einer beliebigen Gruppe*, Comment. Math. Helv. **18** (1945–46), 232–282.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algèbrique*, Part I: 4 (1960); Part II: 8(1961); Part III: 11 (1961), 17 (1963); Part IV: 20 (1964), 24 (1965), 28 (1966), 32 (1967), Publ. I. H. E. S..

[EM42] S. Eilenberg and S. Mac Lane, *Group extensions and homology*, Annals of Math. **43** (1942), 757–831.

[EM43] S. Eilenberg and S. Mac Lane, *Relations between homology and homotopy groups*, Proc. Nat. Acad. Sci. **29** (1943), 155–158; *Relations between homology and homotopy groups of spaces*, Annals of Math. **46** (1945), 480–509.

[EM45] S. Eilenberg and S. Mac Lane, *Natural isomorphisms in group theory*, Proc. Nat. Acad. Sci. **28** (1942); *General theory of natural equivalences*, Trans. AMS **58** (1945), 231–294.

[EM47] S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. I*, Annals of Math. **48** (1947), 51–78.

[EM54] S. Eilenberg and S. Mac Lane, *On the groups $H(\Pi, n)$. II*, Annals of Math. **60** (1954), 49–139.

[EM62] S. Eilenberg and J. Moore, *Limits and spectral sequences*, Topology **1** (1962), 1–23.

[EM66] S. Eilenberg and J. Moore, *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors*, Comment. Math. Helv. **40** (1966), 199–236.

[ES45] S. Eilenberg and N. Steenrod, *Axiomatic approach to homology theory*, Proc. Nat. Acad. Sci. USA **31** (1945), 117–120.

[ES] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton U. Press, Princeton, 1952.

[EZ50] S. Eilenberg and J. Zilber, *Semi-simplicial complexes and singular homology*, Annals of Math. **51** (1950), 499–513.

[EZ53] S. Eilenberg and J. Zilber, *On products of complexes*, Amer. J. Math. **75** (1953), 200–204.

[F36] H. Fitting, *Über den Zusammenhang zwischen dem Begriff der Gleichartigkeit zweier Ideale und dem Äquivalenzbegriff der Elementarteilertheorie*, Math. Annalen **112** (1936), 572–582.

[F46] H. Freudenthal, *Der Einfluss der Fundamental Gruppe auf die Bettischen Gruppen*, Annals of Math. **47** (46), 274–316.

[F64] J. P. Freyd, *Abelian Categories*, Harper & Row, 1964.

[FT] B. Feigin and B. Tsygan, *Cohomology of Lie algebras of generalized Jacobi matrices*, Funkt. Anal. Appl. **17** (1983 86–87); Funct. Anal. Appl. **17** (1983), 153–155.

[G57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **9** (1957), 119–221.

[G62] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–428.

[G64] M. Gerstenhaber, *On the deformation of rings and algebras*, Annals of Math. **79** (1964), 59–103; II, Annals of Math. **84** (1966), 1–19.

[G65] J. Giraud, *Cohomologie non abélienne*, C. R. Acad. Sci. (Paris) **260** (1965), 2666–2668.

[G67] A. Grothendieck, *Local Cohomology*, Lecture Notes vol. 41, Springer Verlag, 1967.

[G71] S. Gersten, *On the functor K_2*, J. Algebra **17** (1971), 212–237.

[G86] T. Goodwillie, *Relative algebraic K-theory and cyclic homology*, Annals of Math. **24** (1986), 347–402.

[GAGA] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–56), 1–42.

[GFGA] A. Grothendieck, *Fondements de la géométrie algébrique*, Extraits du Séminaire Bourbaki, 1957–1962, Secrétariat mathématique, Paris, 1962.

[Gir63] J. Giraud, *Analysis situs*, Séminaire Bourbaki 1962/63, No. 256, 1964.

[Gode] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, 1958.

[H38] M. Hall, *Group rings and extensions, I*, Annals of Math. **39** (1938), 220–234.

[H56] M. Harada, *Note on the dimension of modules and algebras*, J. Inst. Poly. Osaka Univ. Ser. A **7** (1956), 17–27.

[H58] A. Heller, *Homological algebra in abelian categories*, Annals of Math. **68** (1958), 484–525.

[H62] D. K. Harrison, *Commutative algebras and cohomology*, Trans. AMS **104** (1962), 191–204.

[H66] R. Hartshorne, *Residues and Duality*, Springer Lecture Notes vol. 20, Springer Verlag, 1966.

[Hf41] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Annals of Math. **42** (1941), 22-52.

[Hf42] H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1941–42), 257–309; announced, *Relations between the fundamental group and the second Betti Group*, Lectures in Topology, Ann Arbor, 1942, pp. 315–316.

[Hf44] H. Hopf, *Über die Bettischen Gruppen, die zu einer beliebigen Gruppe gehören*, Comment. Math. Helv. **17** (1944–45), 39–79.

[Hh42] G. Hochschild, *Semisimple algebras and generalized derivations*, Amer. J. Math. **64** (1942), 677–694.

[Hh45] G. Hochschild, *On the cohomology groups of an associative algebra*, Annals of Math. **46** (1945), 58–67.

[Hh50] G. Hochschild, *Local class field theory*, Annals of Math. **51** (1950), 331–347.

[Hilb] D. Hilbert, *Über die Theorie der Algebraischen Formen*, Math. Annalen **36** (1890), 473–534.

[HKR] G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. AMS **102** (1962), 383–408.

[HK71] J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen-Macauley-Rings*, Lecture Notes vol. 238, Springer Verlag, 1971.

[HR76] M. Hochster and J. Roberts, *The purity of the Frobenius and local cohomology*, Advances in Math. **21]yr1976**, 117–172..

[HS53] G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans AMS **74** (1953), 110–134; *Cohomology of Lie algebras*, Annals of Math. **57** (1953), 591—603.

[HN52] G. Hochschild and T. Nakayama, *Cohomology in class field theory*, Annals of Math. **55** (1952), 348–366.

[Hö] O. Hölder, *Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4*, Math. Ann. **43** (1893), 301–412.

[HStm] P. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer Verlag, 1971.

[Hu36] W. Hurewicz, *Beitrage zur Topologie der Deformationen. IV. Asphärische Räume*, Proc. Akad. Amsterdam **39** (1936), 215–224.

[Hu41] W. Hurewicz, *On duality theorems*, abstract 47-7-329, Bull. AMS **47** (1941), 562–563.

[J37] N. Jacobson, *Abstract derivation and Lie algebras*, Trans AMS **42** (1937), 206–224.

[JP91] M. Jibladze and T. Pirashvili, *Cohomology of algebraic theories*, J. Algebra **137** (1991), 253–296.

[K23] H. Künneth, *Über die Bettischen Zahlen einer Produktmannigfaltigkeit*, Math. Ann. **90** (1923), 65–85; *Über die Torsionszahlen von Produktmannigfaltigkeiten*, Math. Ann. **91** (1924), 125–134.

[K47] J.-L. Koszul, *Sur les opérateurs de dérivation dans un anneau*, C. R. Acad. Sci. (Paris) **224** (1947), 217–219; *Sur l'homologie des espaces homogènes*, 477-479.

[K50] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127.

[K52] I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. AMS **72** (1952), 327–340.

[KT55] Kawada and J. Tate, *On the Galois cohomology of unramified extensions of function fields in one variable.*, Amer. J. Math. **77** (1955), 197–217.

[Kan56] D. Kan, *Abstract homotopy. III, IV*, Proc. Nat. Acad. Sci. **42** (1956), 419–421, 542–544.

[Kan58] D. Kan, *On functors involving c.s.s. complexes*, Trans. AMS **87** (1958), 330–346.

[K58] I. Kaplansky, *Projective modules*, Annals of Math. **68** (1958), 372–377.

[K59] I. Kaplansky, *Homological dimension of rings and modules*, Univ. of Chicago Dept. of Math., 1959, reprinted as part III of "Fields and Rings," Univ. of Chicago Press, 1969.

[K71] F. Keune, *Algèbre homotopique et K-théorie algébrique*, C.R. Acad. Sci. Paris **273** (1971), A592–A595; *Derived functors and algebraic K-theory*, Lecture Notes in Math. vol. 341, Springer Verlag, 1973, pp. 166–178.

[KP] J. Kelley and E. Pritcher, *Exact homomorphism sequences in homology theory*, Annals of Math. **48** (1947), 682–709.

[L33] S. Lefschetz, *On singular chains and cycles*, Bull. AMS **39** (1933), 124–129.

[L46a] J. Leray, *L'anneau d'homologie d'une représentation*, C. R. Acad. Sci. (Paris) **222** (1946), 1366–1368.

[L46b] ———, *Stucture de l'anneau d'homologie d'une représentation*, C.R. Acad. Sci. (Paris) **222** (1946), 1419–1422.

[L50] ———, *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. Pures Appl. **29** (1950), 1–139.

[L60] S. Lubkin, *Imbedding of abelian categories*, Trans. AMS **97** (1960), 410–417.

[L81] J.-L. Loday, *Symboles en K-théorie algébrique supérieure*, C. R. Acad. Sci. Paris **292** (1981), 863–866.

[LQ] J.-L. Loday and D. Quillen, *Homologie cyclique et homologie de l'algèbre de Lie des matrices*, C. R. Acad. Sci. Paris **296** (1983), 295–297; *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 569–591.

[LS] S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. AMS **128** (1967), 41–70.

[LT] S. Lang and J. Tate, *Principal homogeneous spaces over abelian varieties*, Amer. J. Math **80** (1958), 659–684.

[Lyn] R. Lyndon, *The cohomology theory of group extensions*, Duke Math. J. **15** (1948), 271–292.

[M29] L. Mayer, *Über Abstakte Topologie. I und II*, Monatsh. Math u. Physik **36** (1929), 1–42, 219–258.

[M48] S. Mac Lane, *Groups, categories and duality*, Proc. Nat. Acad. Sci. **34** (1948); *Duality for groups*, Bull. AMS **56** (1950), 485–516.

[M52] W. Massey, *Exact couples in algebraic topology*, Annals of Math. **56** (1952), 363–396.

[M56] S. Mac Lane, *Homologie des anneaux et des modules*, Colloque de topologie algèbrique, Georges Thon, Liège, 1956, pp. 55–80.

[M58] E. Matlis, *Injective modules over noetherian rings*, Pac. J. Math. **8** (1958), 511–528.

[M62] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341.

[M88] S. Mac Lane, *Group extensions for 45 years*, Math. Intelligencer **10** (1988), 29–35..

[M89] S. Mac Lane, *The Applied Mathematics Group at Columbia in World War II*, A Century of Mathematics in America (P. Duren, ed.), vol. III, AMS, 1989, pp. 495–515.

[May] J. P. May, *Stable algebraic topology*, The History of Topology (I. M. James, ed.), 1998.

[MHC] W. Massey, *A history of cohomology theory*, The History of Topology (I. M. James, ed.), 1998.

[ML] S. Mac Lane, *Homology*, Springer-Verlag, 1963.

[Mit] B. Mitchell, *Rings with several objects*, Amer. J. Math **8** (1972), 1–161.

[N25] E. Noether, *Ableitung der Elementarteilertheorie aus der Gruppentheorie*, Nachrichten der 27 Januar 1925, Jahresbericht Deutschen Math. Verein. (2. Abteilung) **34** (1926), 104.

[N58] M. Nagata, *A general theory of algebraic geometry over Dedekind domains. II. Separably generated extensions and regular local rings*, Amer. J. Math. **80** (1958), 382–420.

[N61] Y. Nakai, *On the theory of differentials in commutative rings*, J. Math. Soc. Japan **13** (1961), 63–84.

[O68] B. Osofsky, *Homological dimension and the continuum hypothesis*, Trans. AMS **132** (1968), 217–230.

[Poin] H. Poincaré, *Œvres*, edited by R. Garnier and J. Leray, vol. 6, Gauthiers-Villars, Paris, 1953.

[P35] L. Pontrjagin, *Sur les nombres de Betti des groupes de Lie*, C. R. Acad. Sci. (Paris) **200** (1935), 1277–1280; *Homologies in compact Lie groups*, Mat. Sbornik **6** (1939), 389–422.

[P62] D. Puppe, *On the formal structure of stable homotopy theory*, Colloquium on algebraic topology, Aarhus University, 1962, pp. 65-71.

[PW92] T. Pirashvili, F. Waldhausen, *Mac Lane homology and topological Hochschild homology*, J. Pure Appl. Algebra **82** (1992), 81-98.

[Q67] D. Quillen, *Homotopical Algebra*, Lecture Notes vol. 43, Springer Verlag, 1967.

[Q70] D. Quillen, *On the (co)homology of commutative rings*, Proc. Sympos. Pure Math., vol. XVII, 1970, pp. 65–87.

[Q71] D. Quillen, *Cohomology of groups*, Proc. I. C. M. Nice 1970, 1971, pp. 47–51.

[Q76] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976).

[Riem] B.Riemann, *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, edited by H. Weber, Teubner, Leipzig, 1876.

[R61] J. E. Roos, *Sur les foncteurs dérivés de lim*, C. R. Acad. Sci. (Paris) **252** (1961), 3702–3704.

[R63] G. Rinehart, *Differential forms on general commutative algebras*, TransAMS **108** (1963), 195–222.

[R79] J. E. Roos, *Relations between the Poincaré-Betti series of loop spaces and of local rings*, Lecture Notes in Math. vol. 740, Springer Verlag, 1979, pp. 285–322.

[Rot] J. Rotman, *Notes on Homological Algebra*, Van Nostrand, 1970; *An Introduction to Homological Algebra*, Academic Press, 1979.

[S04] I. Schur, *Über die Darstellungen der endlichen Gruppen durch gegebene lineare Substitutionen*, J. Reine Angew. Math. **127** (1904), 20–50.

[S26] O. Schreier, *Über die Erweiterungen von Gruppen. I*, Monatsh. Math. u. Phys. **34** (1926), 165–180.

[S36] N. Steenrod, *Universal homology groups*, Amer. J. Math. **58** (1936), 661-701.

[S40] N. Steenrod, *Regular cycles of compact metric spaces*, Annals of Math. **41** (1940), 833–851.

[S67] L. Smith, *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. AMS **129** (1967), 58–93.

[S70] R. Sharp, *Local cohomology theory in commutative algebra*, Quart. J. Math. **21** (1970), 425–434.

[S71] R. Sharp, *On Gorenstein modules over a complete Cohen-Macaulay local ring*, Quart. J. Math. **22** (1971), 425–434.

[S76] A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), 1063–1066; Soviet Math. Dokl. **17** (1976), 1160–1164.

[S88] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.

[Se50] J.-P. Serre, *Cohomologie des extensions de groupes*, C. R. Acad. Sci. (Paris) **231** (1950), 643–646.

[Se51] J.-P. Serre, *Homologie singulière des espaces fibrés*, Annals of Math. **54** (1951), 425–505.

[Se55] J.-P. Serre, *Faisceaux algébrique cohérents*, Annals of Math. **61** (1955), 197–278.

[Se56] J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proc. Sympos. Algebraic Number Theory 1955, Science Council of Japan, Tokyo, 1956, pp. 175–189.

[Se58] J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, Exposé 23, Seminaire P. Dubreil 1957/58, Secrétariat mathématique, Paris, 1958.

[SeCG] ———, *Cohomologie Galoisienne*, Lecture Notes in Math. vol. 5, Springer Verlag, 1964.

[SeM] ———, *Algèbre Locale Multiplicités*, Mimeographed Notes, I.H.E.S., 1957; Lecture Notes in Math. vol. 11, Springer Verlag, 1965.

[SGA4] M. Artin, A. Grothendieck and J.-L. Verdier, *Théorie des Topos et Cohomologie Etale des Schémas*, Lecture Notes in Math. vols. 269, 270, 305, Springer Verlag, 1972–73.

[SGA6] P. Berthelot, A. Grothendieck and L. Illusie, *Théorie des Intersections et Théorème de Riemann-Roch*, Lecture Notes in Math. vol. 225, Springer Verlag, 1971.

[Shuk] U. Shukla, *Cohomologie des algèbres associatives*, Ann. Sci. Éc. Norm. Sup. **78** (1961), 163–209.

[Sw59] R. Swan, *Projective modules over finite groups*, Bull. AMS **65** (1959), 365–367.

[Sw62] R. Swan, *Vector bundles and projective modules*, Trans. AMS **105** (1962), 264–277.

[Sw70] R. Swan, *Nonabelian homological algebra and K-theory*, Proc. Sympos. Pure Math., vol. XVII, 1970, pp. 88–123.

[Sw72] R. Swan, *Some relations between higher K-functors*, J. Alg. **21** (1972), 113–136.

[T52] J. Tate, *The higher dimensional cohomology groups of class field theory*, Annals of Math. **56** (1952), 294–297.

[T54] J. Tate, *The cohomology groups of algebraic number fields*, Proc. I. C. M. Amsterdam 1954, North Holland, 1954, pp. 66–67.

[T57] J. Tate, *Homology of noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27.

[T83] B. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology. (Russian)*, Uspekhi Mat. Nauk **38** (1983), 217–218; Russian Math. Surveys **38** (1983), 198–199.

[V] J.-L. Verdier, *Catégories dérivées, état 0*, SGA $4\frac{1}{2}$, Lecture Notes in Math. vol. 569, Springer Verlag, 1977.

[VA] O. Veblen and J. Alexander, *Manifolds of n dimensions*, Annals of Math. **14** (1913), 163–178.

[W36] J. H. C. Whitehead, *On the decomposition of an infinitesimal group*, Proc. Camb. Phil. Soc. **32** (1936), 229–237; *Certain equations in the algebra of a semi-simple infinitesimal group*, Quart. J. Math. **8** (1937), 220–237.

[W37] H. Whitney, *On products in a complex*, Proc. Nat. Acad. Sci. **23** (1937), 285–291; Annals of Math. **39** (1938), 397–432.

[W38] H. Whitney, *Tensor products of abelian groups*, Duke Math. J. **4** (1938), 495–528.

[W78] F. Waldhausen, *Algebraic K-theory of spaces I*, Proc. Symp. Pure Math., vol. 32, 1978, pp. 35–60.

[Y54] N. Yoneda, *On the homology theory of modules*, J. Fac. Sci. Tokyo **7** (1954), 193–227.

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