Planar Graphs

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Rutgers University
Swastik Kopparty

A graph is called planar if it can be drawn in the plane ($\mathbb{R}^2$) with vertex v drawn as a point $f(v) \in \mathbb{R}^2$, and edge (u, v) drawn as a continuous curve between $f(u)$ and $f(v)$, such that no two edges intersect (except possibly at the end-points).

Given a planar drawing of a planar graph, in addition to the vertices and edges of the drawing, we also have **faces** of the drawing. The faces are the connected components of $\mathbb{R}^2$ after we delete the vertices and edges of the drawing.

One can find many examples of planar graphs by trial and error. Every tree is planar. Every cycle C_n is planar. K_4 is planar. But a naive attempt at drawing K_5 planarly fails (Try it). In fact, K_5 is not planar. Showing this is not a joke (just because your attempt to draw it planarly failed doesn't mean it is not planar). We will show that K_5 is not planar by first deriving some property of all planar graphs which does not refer to the planar embedding, and then showing that K_5 does not satisfy this property.

1 Euler's Formula

Theorem 1. *Let $G = (V, E)$ be a connected planar graph. Let F be the set of faces of a planar drawing of G. Then*

$$|V| - |E| + |F| = 2.$$

Proof. By induction.

If G is acyclic, then $|F| = 1$, and the theorem holds because then G is a tree and $|E| = |V| - 1$.

Otherwise G has a cycle. Let e be an edge in a cycle. Deleting e from G and its planar drawing results in a graph G' with $|V|$ vertices, $|E| - 1$ edges and $|F| - 1$ faces (since deleting an edge involved in a cycle merges the two faces on either side of it).

By induction, we have that $|V| - (|E| - 1) + (|F| - 1) = 2$, and so $|V| - |E| + |F| = 2$. □

The homework has a more illuminating proof of Euler's formula.

By itself, Euler's formula does not provide us with a tool for showing that some graphs are not planar, because it refers to the set of faces F in a potential planar drawing of the graph.

But we can use it to derive the following strong sufficient criterion for nonplanarity.

Lemma 2. *If $G = (V, E)$ is a connected planar graph and $|V| > 2$, then $|E| \leq 3|V| - 6$.*

Proof. If $|E| \leq 3$, then the theorem can be verified by inspection (note that $|V| > 2$).

Otherwise, since there are at least 3 edges, every face has at least 3 edges bounding it. Furthermore, every edge bounds at most 2 faces. Thus, $3|F| \geq 2|E|$.

Taking this relation and substituting it into Euler's formula, we get: $|E| \leq 3|V| - 6$. $\square$

Corollary 3. *The average degree of vertices in a planar graph is strictly less than* 6.

In fact, the same argument shows that if a planar graph has no small cycles, we can get even stronger bounds on the number of edges (in the extreme, a planar graph with no cycles at all is a tree and has at most $|V| - 1$ edges).

Lemma 4. *If $G = (V, E)$ is a planar graph with $|E| \geq g$ and no cycle of length $< g$, then:*

$$|E| \leq \frac{g}{g-2}(|V| - 2).$$

Proof. The proof is identical to the previous proof, except we now know that every face has at least g edges on it (we need $|E| \geq g$ for this).

Thus $g|F| \geq 2|E|$, and so by Euler's formula: $|E| \geq \frac{g}{g-2}(|V| - 2)$. $\square$

2 Some non-planar graphs

We now use the above criteria to find some non-planar graphs.

- K_5: K_5 has 5 vertices and 10 edges, and thus by Lemma 2 it is not planar.
- $K_{3,3}$: $K_{3,3}$ has 6 vertices and 9 edges, and so we cannot apply Lemma 2. But notice that it is bipartite, and thus it has no cycles of length 3. We may apply Lemma 4 with $g = 4$, and this implies that $K_{3,3}$ is not planar.
- Any graph containing a nonplanar graph as a subgraph is nonplanar. Thus K_6 and $K_{4,5}$ are nonplanar.
- In fact, any graph which contains a "topological embedding" of a nonplanar graph is nonplanar. A topological embedding of a graph H in a graph G is a subgraph of G which is isomorphic to a graph obtained by replacing each edge of H with a path (with the paths all vertex disjoint).

An absolutely stunning fact is that these observations capture all nonplanar graphs! The nonplanarity of the specific graphs K_5 and $K_{3,3}$ was a very old fact. Little did they know back then how fundamental these examples were. This is Kuratowski's theorem from 1935[1].

Theorem 5. *A graph G is planar if and only if it contains a topological embedding of K_5 or a topological embedding of $K_{3,3}$.*

We will not prove Kuratowski's planarity criterion, but we will see an efficient algorithm for detecting whether a graph is planar, and how to draw a planar graph planarly.

[1] As a side note, Kuratowski was a giant of general topology and set theory (with very very large sets). Kuratowski proved "Zorn's Lemma" first, 20 years before Zorn had anything to do with it. It is amazing that he descended down to finite planar graph theory and gave it such a gem.

3 Coloring Planar Graphs

One of the major stimulants for the study of planar graphs back in the 1800s was the 4-color conjecture.

Given a planar graph, how many colors do you need in order to color the vertices so that no two adjacent vertices get the same color (this can also be phrased in the language of coloring regions of a geographic map so that no adjacent regions get the same color). Long ago, Francis Guthrie conjectured that 4 colors suffice. This became a very notorious problem.

We will eventually see that 5 colors suffice.

Theorem 6 (6-color theorem)**.** *Every planar graph G can be colored with 6 colors.*

Proof. By induction on the number of vertices in G.

By Corollary 3, G has a vertex v of degree at most 5. Remove v from G. The remaining graph is planar, and by induction, can be colored with at most 6 colors. Now bring v back. At least one of the 6 colors is not used on the ≤ 5 neighbors of v. Thus, we may color v one of the unused colors, and this gives a valid 6-coloring of G. □

In 1890, Heawood brought the first serious ideas to this problem, and proved that planar graphs could be 5 colored (along the way, he found a flaw in Kempe's 11 year old widely accepted "proof" of the 4-color conjecture).

Theorem 7 (5-color theorem)**.** *Every planar graph G can be colored with 5 colors.*

Proof. If G has a vertex of degree ≤ 4, then we are done by induction as in the previous proof.

If not, by Corollary 3, G has a vertex v of degree 5. Remove v from G. The remaining graph is planar, and by induction, can be colored with at most 5 colors. Now bring v back.

If among the 5 neighbors of v in G, not all 5 colors are used, then we may color v the missing color, and we are done.

Otherwise, we are in the following situation. v has 5 neighbors $v_1, \ldots, v_5$, and v_i is colored with color i. By renaming, we may assume that at v the edges from v to $v_1, \ldots, v_5$ are arranged in clockwise order.

We need to provide color for v. To do this, we necessarily have to modify the coloring of the remaining graph.

First let us explore the possibility of replacing the color of v_1 with color 3. To do this, consider the graph H induced on the vertices colored 1 and 3.

- **Case 1:** If v_1 is disconnected from v_3 in H, then consider the component of H containing v_1, and swap the colors 1 and 3 on the component.
- **Case 2:** Otherwise, there is a path from v_1 to v_3 in H. This path, along with the edges $\{v, v_1\}$ and $\{v, v_3\}$ forms a **Jordan Curve** separating vertex v_2 from vertex v_4. Thus the graph induced on vertices colored 2 and 4 cannot have v_2 and v_4 connected! We may thus do what we did in Case 1 to v_2 and v_4.

By these operations, we now have that there are only 4 colors appearing on $v_1, \ldots, v_5$. We may thus color v with the remaining 5th color. $\square$

In fact, this gives an efficient algorithm to 5-color a planar graph (given its planar drawing).

In the 1970s, Appel and Haken proved the 4-color conjecture. This nonconventional proof was based on two ingredients:

- reduction to verifying the conjecture for a certain finite number of graphs (this is based on work of Heesch from the 1950s),
- and then verifying the conjecture on those graphs via a computer.

Till today, no simple computer-less proof of this theorem is known!