

Disproportionate division

Logan Crew, Bhargav Narayanan, and Sophie Spirkl

ABSTRACT. We study the disproportionate version of the classical cake-cutting problem: how efficiently can we divide a cake, here $[0, 1]$, among $n \geq 2$ agents with different demands $\alpha_1, \alpha_2, \dots, \alpha_n$ summing to 1? When all the agents have equal demands of $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1/n$, it is well-known that there exists a fair division with $n - 1$ cuts, and this is optimal. For arbitrary demands on the other hand, folklore arguments from algebraic topology show that $O(n \log n)$ cuts suffice, and this has been the state of the art for decades. Here, we improve the state of affairs in two ways: we prove that disproportionate division may always be achieved with $3n - 4$ cuts, and also give an effective algorithm to construct such a division. We additionally offer a topological conjecture that implies that $2n - 2$ cuts suffice in general, which would be optimal.

1. INTRODUCTION

The cake-cutting problem, whose study was initiated by Banach and Steinhaus [8] in 1949, is a classical measure partitioning problem concerned with the division of a ‘cake’, here the unit interval $[0, 1]$, amongst $n \geq 2$ agents each with their own ‘utilities’, here non-atomic Borel probability measures $\mu_1, \mu_2, \dots, \mu_n$ on $[0, 1]$.

Traditionally, one is interested in *fair divisions*: a partition $X_1 \cup X_2 \dots \cup X_n$ of $[0, 1]$ is said to be a *fair division* if $\mu_i(X_i) \geq 1/n$ for all $1 \leq i \leq n$. That a fair division always exists is a classical fact, and is more or less trivial to see. Slide a knife from 0 to 1, stopping at the first point $x \in [0, 1]$ where $\mu_i([0, x]) \geq 1/n$ for some $1 \leq i \leq n$, set $X_i = [0, x)$, and recurse on $[x, 1]$ with the remaining $n - 1$ agents; this in fact shows us that there is a fair division with $n - 1$ cuts, which is optimal since $n - 1$ cuts are necessary just to partition the unit interval into n pieces. There is now a significant body of work on fair divisions, investigating various aspects of this problem such as envy-free-ness, equitability and computational complexity, and utilising a wide variety of tools from combinatorics, topology and game theory; for a small sample of the literature, see [6, 10, 12, 3, 4, 1].

Our focus here is a natural ‘disproportionate’ generalisation of the fair division problem: what can we say when the n agents have differing claims to the cake? In the measure partitioning literature, the disproportionate problem often presents subtleties

Date: 5 September 2019.

2010 *Mathematics Subject Classification.* Primary 05D05; Secondary 91B32.

not inherent in the proportionate problem (notably in [9, 2], for example), and as we shall shortly see, this is also the case here.

Given non-negative *demands* $\alpha_1, \alpha_2, \dots, \alpha_n$ summing to 1, a *disproportionate division* for these demands is a partition $X_1 \cup X_2 \cdots \cup X_n$ of $[0, 1]$ with $\mu_i(X_i) \geq \alpha_i$ for all $1 \leq i \leq n$. The requirement in this definition that all the demands sum to 1 is sensible since the measures $\mu_1, \mu_2, \dots, \mu_n$ are, in general, allowed to be identical.

Disproportionate divisions are easily seen to exist when all the demands are rational: bring all the demands to a common denominator, say D , construct a fair division for D agents, and then distribute the resulting pieces appropriately. However, this does not immediately demonstrate existence for irrational demands, since taking limits with respect to the Hausdorff metric, for example, does not preserve measure. Nonetheless, it can be shown using an infinite analogue of the sliding knife procedure that disproportionate divisions exist for all demands, though when proceeding thusly, one cannot in general avoid ‘crumbs’: the pieces of cake produced by this argument are only guaranteed to be countable unions of intervals.

What if we ask for an *efficient* disproportionate division with a *bounded* number of cuts? A beautiful topological result of Stromquist and Woodall [11] furnishes an answer; a folklore argument utilising this result shows that a disproportionate division for n agents with arbitrary demands may always be found with $O(n^2)$ cuts, and a more efficient rendition of this argument, recently discovered by Segal-Halevi [7], shows that in fact $O(n \log n)$ cuts always suffice.

Our main result improves on these decades-old topological arguments and shows that linearly many cuts are sufficient (and of course necessary) in general.

Theorem 1.1. *For all $n \geq 2$, given non-atomic probability measures $\mu_1, \mu_2, \dots, \mu_n$ on $[0, 1]$ and non-negative reals $\alpha_1, \alpha_2, \dots, \alpha_n$ summing to 1, there exists a disproportionate division for these demands with at most $3n - 4$ cuts.*

Again, it is clear that $n - 1$ cuts are always necessary for n agents, so this result is tight up to multiplicative constants. Our proof of Theorem 1.1 is combinatorial as opposed to topological; an attractive byproduct of this approach is that the proof is constructive, yielding an effective algorithm for disproportionate division.

While Theorem 1.1 determines the optimal number of cuts for disproportionate division with n agents up to multiplicative constants, the problem of pinning down this extremal number still remains. Unlike with fair division, it turns out that $n - 1$ cuts do not always suffice; there is a construction demonstrating that $2n - 2$ cuts may be necessary in general. We suspect that this construction, not Theorem 1.1, reflects the truth, and that the tightness of this construction should follow from topological considerations: to this end, we shall present a topological conjecture that implies that $2n - 2$ cuts always suffice for disproportionate division with n agents.

This paper is organised as follows. We give the proof of Theorem 1.1 in Section 2. We conclude by discussing lower bounds, and by presenting a measure partitioning conjecture implying optimal bounds, in Section 3.

2. PROOF OF THE MAIN RESULT

In this section, we prove our main result. It will be helpful to have some notation. As is usual, we write $[n]$ for the set $\{1, 2, \dots, n\}$. Let us also write $f(n)$ for the maximum number of cuts needed (over all possible measures and demands) for disproportionate division with n agents; as noted earlier, $f(n)$ exists for all $n \in \mathbb{N}$ and we have the estimate $f(n) = O(n \log n)$. Recall that our result, in this language, asserts that $f(n) \leq 3n - 4$ for all $n \geq 2$.

Before we turn to the proof, let us briefly highlight the main idea behind the proof. We shall proceed by induction on n . A natural first attempt is as follows. As in the sliding knife procedure, make $O(1)$ cuts to

- partition the unit interval $[0, 1]$ into two pieces A and B , and
- find a suitable partition of the n agents into two groups of size a and b with $a + b = n$,

so that we may recursively solve the division problem for the first group of size a on A and the second group of size b on B . This would yield the estimate $f(n) \leq f(a) + f(b) + O(1)$, which would in turn imply that $f(n) = O(n)$. While almost all the arguments in the literature (that we are aware of) follow this rough outline, we have however been unable to find a proof along these lines. Instead, it turns out to be quite helpful to allow ourselves a little more elbow room: once we have A and B , instead of attempting to partition the n agents into two groups (thereby generating two subproblems whose sizes add up to n), we shall allow some agents to participate in both subproblems (i.e., on both A and B), thereby producing subproblems whose sizes add up to more than n ; we shall then make up for this inefficiency by ensuring that these subproblems are both suitably smaller than n .

Proof of Theorem 1.1. First, a matter of convenience: since all our measures are non-atomic, open, half-open and closed intervals with the same endpoints all have the same measure in each of our measures, so we shall use these interchangeably as appropriate.

Let us record two simple facts to start with. We trivially have $f(1) = 0$. Next, note that $f(2) \leq 2$, which we may infer from the following fact: given two non-atomic probability measures μ and μ' on the unit circle S^1 and any $\alpha \in [0, 1]$, there is an interval $X \subset S^1$ such that $\mu(X) = \alpha$ and $\mu'(X) \geq \alpha$. To see this, note that, by compactness, it suffices to prove the claim for rational α ; indeed, the limit of a sequence of closed intervals with respect to the Hausdorff metric is a closed interval. Writing $\alpha = p/q$, we may find intervals X_1, X_2, \dots, X_q so that $\mu(X_i) = \alpha$ for each $1 \leq i \leq q$, and so each

point of the circle is covered by exactly p of these intervals; by pigeonholing, it is now clear that $\mu'(X_i) \geq \alpha$ for some $1 \leq i \leq q$. Alternately, a different (albeit ineffective) way of seeing this is to appeal to the Stromquist–Woodall theorem [11] for two measures, which in fact promises the existence of an interval X with $\mu(X) = \mu'(X) = \alpha$.

With the above bounds for $f(1)$ and $f(2)$ in hand, we shall establish the following estimate: for each $n \geq 2$,

$$f(n+1) \leq \max\{1+f(n), \max_{2 \leq k \leq n} \{1+f(k)+f(n+2-k)\}\} \leq \max_{2 \leq k \leq n} \{1+f(k)+f(n+2-k)\}. \quad (1)$$

We shall prove the first inequality above; the second follows from the trivial fact that $1+f(n) \leq 1+f(2)+f(n)$, and the result then easily follows, by induction, from this second inequality.

We prove the above estimate as follows. Given measures $\mu_1, \mu_2, \dots, \mu_{n+1}$ and demands $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ with $n \geq 2$, define for $\vartheta \in [0, 1]$ and $k \in [n+1]$, the set

$$S(\vartheta, k) = \{i : \mu_i([0, \vartheta]) > 1/2\} \cup \{i : \mu_i([0, \vartheta]) = 1/2 \text{ and } i \leq k\}.$$

Now, fix a minimal $x \in [0, 1]$ and then a minimal $t \in [n+1]$ so that

$$\sum_{i \in S(x, t)} \alpha_i \geq 1/2.$$

Next, set $P = S(x, t) \setminus \{t\}$ and $Q = [n+1] \setminus S(x, t)$, whence $[n+1] = P \cup Q \cup \{t\}$. By the minimality of x and t , note that $\mu_i([0, x]) \geq 1/2$ for all $i \in P$, $\mu_i([0, x]) \leq 1/2$ for all $i \in Q$, and $\mu_t([0, x]) = 1/2$; furthermore, we also have $\sum_{i \in P} \alpha_i \leq 1/2$ and $\sum_{i \in Q} \alpha_i \leq 1/2$.

If both P and Q are non-empty, then we are done. Indeed, set $\alpha' = 1/2 - \sum_{i \in P} \alpha_i$ and $\alpha'' = 1/2 - \sum_{i \in Q} \alpha_i$ so that $\alpha', \alpha'' \geq 0$ and $\alpha' + \alpha'' = \alpha_t$. Now, note that the division problem on $[0, x]$ for the measures in $P \cup \{t\}$, with the original demands α_i for each measure μ_i with $i \in P$, and demand α' for the measure μ_t , is (after rescaling by constants) a smaller instance of the disproportionate division problem for $k = |P| + 1$ measures, with $2 \leq k \leq n$. Similarly, the division problem on $(x, 1]$ for the measures in $Q \cup \{t\}$, with the original demands α_i for each measure μ_i with $i \in Q$, and demand α'' for the measure μ_t , is again a smaller instance of the disproportionate division problem for $n+2-k$ measures. By making a cut at x and solving these two smaller division problems, we see that (1) holds in this case.

Now, assume that one of P or Q is empty. We claim that it must have been that $\alpha_t \geq 1/2$. Indeed, if $P = \emptyset$, then $S(x, t) = \{t\}$ and the claim follows from the definition of $S(x, t)$, and if $Q = \emptyset$, then $\alpha_t = 1 - \sum_{i \in P} \alpha_i \geq 1 - 1/2 = 1/2$.

If $\alpha_t = 1/2$, then we are again done. Indeed, if $P = \emptyset$, then assign the piece $[0, x]$ to μ_t ; having made one cut at x , we are now left with the division problem on $(x, 1]$ for

the n measures μ_i with $i \in [n+1] \setminus \{t\}$, establishing (1). If $Q = \emptyset$, then we proceed similarly by assigning the piece $(x, 1]$ to μ_t .

We are left to address the case where $\alpha_t > 1/2$ and one of P or Q is empty. We now describe how to establish (1) when $P = \emptyset$; the other case may be handled analogously by exchanging the roles of P and Q .

Since $P = \emptyset$, note that $\mu_t([0, x]) = 1/2$ and $\mu_i([0, x]) \leq 1/2$ for each $i \neq t$. We know that $\alpha_t > 1/2$, so we may fix a minimal $y \in (x, 1]$ such that $\mu_t([0, y]) = \alpha_t$, and define $U = \{i : \mu_i([0, y]) \geq \alpha_t \text{ and } i \neq t\}$ and $V = \{i : \mu_i([0, y]) < \alpha_t \text{ and } i \neq t\}$, so that U and V partition $Q = [n+1] \setminus \{t\}$.

If $U = \emptyset$, we are done as before by cutting at y , assigning $[0, y]$ to μ_t and recursively solving the division problem on $(y, 1]$ for the remaining measures. We may therefore assume that U is non-empty. If V is also non-empty, we may finish as before by cutting at y and recursively solving two smaller division problems: the division problem on $[0, y]$ for the measures in $U \cup \{t\}$, with the original demands α_i for each measure μ_i with $i \in U$, and demand $\alpha_t - \sum_{i \in U} \alpha_i$ for the measure μ_t , and the division problem on $(y, 1]$ for the measures in $V \cup \{t\}$, with the original demands α_i for each measure μ_i with $i \in V$, and demand $1 - \alpha_t - \sum_{i \in V} \alpha_i$ for the measure μ_t .

Thus, we are left to deal with the case where $U = Q = [n+1] \setminus \{t\}$ and $V = \emptyset$. Recall that we have $P = \emptyset$. To summarise, we now know that $\mu_i([0, x]) \leq 1/2$ and $\mu_i([0, y]) \geq \alpha_t$ for each $i \in U$, and that $\mu_t([0, x]) = 1/2$ and $\mu_t([0, y]) = \alpha_t$. Therefore, appealing to the intermediate value theorem, we may choose a maximal $z \in (x, y]$ such that $\mu_s([0, z]) = \mu_t([0, z]) = \beta$ for some $s \in U$; by the definition of z , we have $\mu_i([0, z]) \geq \beta$ for each $i \in U$. Notice that we have $1 - \beta \geq 1 - \alpha_t \geq \alpha_i$ for all $i \neq t$, where the first inequality follows from monotonicity, and the second from the fact that all the demands sum to 1. Writing $U' = U \setminus \{s\}$, we may now establish (1) by cutting at z and recursively solving two smaller division problems: the division problem on $[0, z]$ for the measures in $U' \cup \{t\}$, with the original demands α_i for each measure μ_i with $i \in U'$, and demand $\beta - \sum_{i \in U'} \alpha_i$ for the measure μ_t (which is a non-negative demand since $\beta \geq 1/2$ and $\sum_{i \in U'} \alpha_i \leq \sum_{i \in U} \alpha_i \leq 1/2$), and the division problem on $(z, 1]$ for the measures in $\{s, t\}$, with the original demand α_s for the measure μ_s , and demand $1 - \alpha_s - \beta$ for the measure μ_t .

We now see that we may establish the estimate (1) in each case; this completes the proof. \square

In the computational complexity literature, cake-cutting algorithms are often represented by a sequence of queries of the form ‘evaluate a given interval in a given measure’ or ‘mark off an interval of a given value in a given measure’, and query-complexity is the object of interest; see [5], for example. We mention in closing that the algorithm described here may also be represented in this form, and this representation is easily

seen to have bounded query-complexity (though we have made no attempt to optimise the query-complexity of our algorithm, since this is not our primary motivation).

3. CONCLUSION

We have shown that $f(n) \leq 3n - 4$ for all $n \geq 2$. While we trivially have $f(n) \geq n - 1$ for $n \geq 2$, it is not hard to do better. The following construction from [7] shows that $f(n) \geq 2n - 2$ for all $n \geq 2$. Let μ_1 be the uniform measure on $[0, 1]$ and for $1 \leq i \leq n - 1$, let μ_{i+1} be the uniform measure supported on the (tiny) interval $[i/n - \varepsilon, i/n + \varepsilon]$, where $\varepsilon = 1/(100n)^{10}$. For the demands $\alpha_1 = 1 - \varepsilon^{10}$ and $\alpha_i = \varepsilon^{10}/(n - 1)$ for each $2 \leq i \leq n$, it is easily seen that any disproportionate division requires at least two cuts in the support of each of the measures $\mu_2, \mu_3, \dots, \mu_n$, for a total of $2n - 2$ cuts.

We suspect that the above construction reflects the truth and that $f(n) = 2n - 2$ for all $n \geq 2$. This conjecture is particularly interesting because we believe there are deeper topological considerations driving this claim. Concretely, the following measure partitioning conjecture, if true, would imply by a simple inductive argument that $f(n) = 2n - 2$ for all $n \geq 2$.

Conjecture 3.1. *For any $n \geq 2$ non-atomic probability measures $\mu_1, \mu_2, \dots, \mu_n$ on the unit circle S^1 and non-negative reals $\alpha_1, \alpha_2, \dots, \alpha_n$ summing to 1, there exists a partition of the set $[n] = P \cup Q$ into two nonempty sets and a partition of the circle $S^1 = X \cup X^c$ into two intervals such that*

$$\min_{i \in P} \mu_i(X) = \sum_{j \in P} \alpha_j \text{ and } \min_{i \in Q} \mu_i(X^c) = \sum_{j \in Q} \alpha_j. \quad (2)$$

We close with a few remarks about this conjecture. First, Conjecture 3.1 for two measures is known to be true and is essentially the Stromquist–Woodall theorem [11] for two measures, but we have unfortunately been unable to decide this conjecture even for three measures. Second, a slightly weaker statement, with inequalities (as opposed to equalities) in (2), would suffice for the purposes of disproportionate division; however, since topological fixed point theorems — the natural tools for such problems — typically yield equalities, we have opted for this slightly stronger conjecture. Finally, our proof of Theorem 1.1 may be seen as establishing an even weaker form of this conjecture, where we obtain a ‘fractional’ partition of the set of measures (i.e., with one measure being distributed ‘fractionally’ across the partition).

ACKNOWLEDGEMENTS

The second author wishes to acknowledge support from NSF grant DMS-1800521, and the third author was supported by NSF grant DMS-1802201. We would like to thank Imre Bárány and Dömötör Pálvölgyi for the many interesting discussions around

the measure partitioning conjecture presented here, and the anonymous referee for several helpful comments.

REFERENCES

1. H. Aziz and S. Mackenzie, *A discrete and bounded envy-free cake cutting protocol for any number of agents*, 57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016, IEEE Computer Soc., Los Alamitos, CA, 2016, pp. 416–427. [1](#)
2. I. Bárány and J. Matoušek, *Simultaneous partitions of measures by k -fans*, Discrete Comput. Geom. **25** (2001), 317–334. [2](#)
3. S. J. Brams, M. A. Jones, and C. Klamler, *Better ways to cut a cake*, Notices Amer. Math. Soc. **53** (2006), 1314–1321. [1](#)
4. S. J. Brams and A. D. Taylor, *An envy-free cake division protocol*, Amer. Math. Monthly **102** (1995), 9–18. [1](#)
5. A. Cseh and T. Fleiner, *The complexity of cake cutting with unequal shares*, Preprint, arXiv:1709.03152. [5](#)
6. L. E. Dubins and E. H. Spanier, *How to cut a cake fairly*, Amer. Math. Monthly **68** (1961), 1–17. [1](#)
7. E. Segal-Halevi, *Cake-cutting with different entitlements: How many cuts are needed?*, Preprint, arXiv:1803.05470. [2](#), [6](#)
8. H. Steinhaus, *Sur la division pragmatique*, Econometrica **17** (1949), 315–319. [1](#)
9. A. H. Stone and J. W. Tukey, *Generalized “sandwich” theorems*, Duke Math. J. **9** (1942), 356–359. [2](#)
10. W. Stromquist, *How to cut a cake fairly*, Amer. Math. Monthly **87** (1980), 640–644. [1](#)
11. W. Stromquist and D. R. Woodall, *Sets on which several measures agree*, J. Math. Anal. Appl. **108** (1985), 241–248. [2](#), [4](#), [6](#)
12. W. A. Webb, *How to cut a cake fairly using a minimal number of cuts*, Discrete Appl. Math. **74** (1997), 183–190. [1](#)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104, USA

Email address: `crew1@math.upenn.edu`

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854, USA

Email address: `narayanan@math.rutgers.edu`

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854, USA

Email address: `sophie.spirkl@rutgers.edu`