Non-tame mouse from the failure of square at a singular strong limit cardinal *†

Grigor Sargsyan‡
Department of Mathematics
Rutgers University,
New Brunswick, NJ, 08854 USA
http://math.rutgers.edu/∼gs481
grigor@math.rutgers.edu

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Abstract

Building on the work of Schimmerling ([8]) and Steel ([14]), we show that the failure of square principle at a singular strong limit cardinal implies that there is a non-tame mouse. The proof presented is the first inductive step beyond $L(\mathbb{R})$ of the core model induction that is aimed at getting a model of $AD_{\mathbb{R}}$ + "Θ is regular" from the failure of square at a singular strong limit cardinal or PFA.

One of the holly grails of inner model program has been to determine the exact consistency strength of forcing axioms such as *Proper Forcing Axiom* (PFA) and *Martin's Maximum* (MM). As the consistency of PFA, MM and other similar axioms have been established relative to one supercompact cardinal, it is natural to conjecture that the exact consistency strength of such forcing axioms is one supercompact cardinal.

Most of the partial results have been obtained by assuming a weakening of PFA, namely the failure of square at various cardinals. For instance, in [14], Steel showed that the failure of square at a singular strong limit cardinal implies that AD holds in $L(\mathbb{R})$. We extend this result further by showing the following theorem.

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Theorem 0.1 (Main Theorem) *Suppose $\neg\Box_\kappa$ holds for some singular strong limit cardinal κ. Then there is M such that $\mathbb{R}, Ord \subseteq M$ and $M \vDash AD^+ + \theta_0 < \Theta$. In particular, there is a non-tame mouse.*

In the statement of Theorem 0.1, θ_0 is the first member of the Solovay sequence (see Definition 2.1). Steel and Woodin showed that the conclusion of Theorem 0.1 implies the existence of a non-tame mouse (see [15]). Our proof of Theorem 0.1 is via the core model induction. It constitutes the first step of the induction that goes beyond the levels of $Lp(\mathbb{R})$[1]. We expect that our proof will generalize to yield a model of $AD_{\mathbb{R}}$ + "Θ is regular".

We advise those readers who are familiar with basic notions of inner model theory to read Section 4. We have added many of the background material needed for this paper in the sections proceeding Section 4. Thus, those readers less familiar with basic definitions can go over the first three sections of the paper before reading Section 4.

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1 Premice, mice and their iteration strategies

We establish some notation and list some basic facts about mice and their iteration strategies. The reader can find more detail in [17]. Suppose $\mathcal{M}$ is a premouse. We then let $o(\mathcal{M}) = Ord \cap \mathcal{M}$. If $\mathcal{M}$ is a premouse and $\xi \leq o(\mathcal{M})$ then we let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (crit(E), lh(E)]$. We say ξ is a *strong cutpoint* if there is no E on $\mathcal{M}$ such that $\xi \in [crit(E), lh(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in $\mathcal{M}$ if η isn't a strong cutpoint of $\mathcal{M}$. Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}_\eta^M = \cup\{\mathcal{N} \lhd \mathcal{M} : \rho(\mathcal{N}) = \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N}\}.$$

If $\mathcal{M}$ is a k-sound premouse, then a *(k, θ)-iteration strategy* for $\mathcal{M}$ is a winning strategy for player II in the iteration game $G_k(\mathcal{M}, \theta)$, and a *k-normal iteration tree* on $\mathcal{M}$ is a play of this

[1] $Lp(\mathbb{R})$ is the stack of all countably iterable sound $\mathbb{R}$-mice projecting to $\mathbb{R}$.

game in which II has not yet lost. (That is, all models are wellfounded.) k-normal trees are called "k-maximal" in [17], but we shall use "maximal" for a completely different property of trees here.) We shall drop the reference to the fine-structural parameter k whenever it seems safe to do so, and speak simply of normal trees.

We say $\mathcal{M}$ is θ-iterable if II has a winning strategy in $G_k(\mathcal{M}, \theta)$. We say $\mathcal{M}$ is *countably* θ-iterable if any countable substructure of $\mathcal{M}$ is θ-iterable. It follows from the copying construction that a θ-iterable mouse is countably θ-iterable. We say $\mathcal{M}$ is *countably* iterable if all of its countable substructures are $\omega_1 + 1$-iterable.

If $\mathcal{T}$ is a normal iteration tree, then $\mathcal{T}$ has the form

$$\mathcal{T} = \langle T, deg, D, \langle E_\alpha, \mathcal{M}_{\alpha+1} | \alpha + 1 < \eta \rangle \rangle.$$

Recall that D is the set of *dropping* points. Recall also that if η is limit then

$$\vec{E}(\mathcal{T}) = \cup_{\alpha<\eta}(\dot{E}^{\mathcal{M}_\alpha \restriction lh(E_\alpha)}),$$
$$\mathcal{M}(\mathcal{T}) = \cup_{\alpha<\eta}\mathcal{M}_\alpha \restriction lh(E_\alpha),$$
$$\delta(\mathcal{T}) = \sup_{\alpha<\eta} lh(E_\alpha).$$

If b is a branch of $\mathcal{T}$ such that $D \cap b$ is finite, then $\mathcal{M}_b^\mathcal{T}$ is the direct limit of the models along b . If $\alpha \leq_T \beta$ and $(\alpha, \beta]_\mathcal{T} \cap D = \emptyset$ then

$$\pi^\mathcal{T}_{\alpha,\beta} : \mathcal{M}^\mathcal{T}_\alpha \to \mathcal{M}^\mathcal{T}_\beta$$

is the iteration map, and if $\alpha \in b$ and $(b - \alpha) \cap D = \emptyset$, then

$$\pi^\mathcal{T}_{\alpha,b} : \mathcal{M}^*_\alpha \to \mathcal{M}^\mathcal{T}_b$$

is the iteration map. If $\mathcal{T}$ has a last model $\mathcal{M}^\mathcal{T}_\alpha$, and the branch $[0, \alpha]_T$ does not drop, then we often write $\pi^\mathcal{T}$ for $\pi^\mathcal{T}_{0,\alpha}$.

Recall that the strategy for a sound mouse projecting to ω is determined by $\mathcal{Q}$-*structures*. For $\mathcal{T}$ normal, let $\Phi(\mathcal{T})$ be the phalanx of $\mathcal{T}$.

Definition 1.1 *Let $\mathcal{T}$ be a k-normal tree of limit length on a k-sound premouse, and let b be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $M^\mathcal{T}_b$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Theorem 1.2 *Let $\mathcal{M}$ be a k-sound premouse such that $\rho_k(\mathcal{M}) = \omega$. Then $\mathcal{M}$ has at most one $(k, \omega_1 + 1)$ iteration strategy. Moreover, any such strategy Σ is determined by: $\Sigma(\mathcal{T})$ is the unique cofinal b such that the phalanx $\Phi(\mathcal{T})^\frown\langle\delta(\mathcal{T}), \deg^{\mathcal{T}}(b), \mathcal{Q}(b, \mathcal{T})\rangle$ is $\omega_1 + 1$-iterable.*

In some cases, however, it is enough to assume that $\mathcal{Q}(b, \mathcal{T})$ is countably iterable. This happens, for instance, when $\mathcal{M}$ has no local Woodin cardinals with extenders overlapping it. While the mice we will consider do have local overlapped Woodin cardinals, the mice themselves will not have such Woodin cardinals. This simplifies our situation somewhat and below we describe exactly how this will be used. We say an iteration tree $\mathcal{T}$ is above η if all the extenders used in $\mathcal{T}$ have critical points $> \eta$.

Definition 1.3 *Suppose $\mathcal{M}$ is a premouse and $\mathcal{T}$ is a normal tree on $\mathcal{M}$. We say $\mathcal{T}$ has a fatal drop if for some α such that $\alpha + 1 < lh(\mathcal{T})$, there is some $\eta < o(\mathcal{M}_\alpha^{\mathcal{T}})$ such that the rest of $\mathcal{T}$ is an iteration tree on $\mathcal{O}_\eta^{\mathcal{M}_\alpha^{\mathcal{T}}}$ above η. Suppose $\mathcal{T}$ has a fatal drop. Let (α, η) be lexicographically least witnessing the fact that $\mathcal{T}$ has a fatal drop. Then we say $\mathcal{T}$ has a fatal drop at (α, η).*

Definition 1.4 (Definition 2.1 of [12]) *Let $\mathcal{T}$ be a normal iteration tree; then $\mathcal{Q}(\mathcal{T})$ is the unique premouse extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1 + 1$-iterable above $\delta(\mathcal{T})$, and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$, if there is any such premouse.*

Countable iterability is enough to guarantee there is at most one premouse with the properties of $\mathcal{Q}(\mathcal{T})$. If it exists, $\mathcal{Q}(\mathcal{T})$ might identify the good branch of $\mathcal{T}$, the one any sufficiently powerful iteration strategy must choose. This is the content of the next lemma which can be proved by analyzing the proof of Theorem 6.12 of [17]

Lemma 1.5 *Suppose $\mathcal{M}$ is a k-sound premouse such that no measurable cardinal of $\mathcal{M}$ is a limit of Woodin cardinals. Suppose $\mathcal{T}$ is a k-normal iteration tree on $\mathcal{M}$ of limit length which doesn't have a fatal drop and suppose $\mathcal{Q}(\mathcal{T})$ exists. Then there is at most one cofinal branch b of $\mathcal{T}$ such that either $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^{\mathcal{T}}$ or $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^{\mathcal{T}}|\xi$ for some ξ in the wellfounded part of $\mathcal{M}_b^{\mathcal{T}}$.*

We shall need to look more closely at what is behind the uniqueness results above, at how the "fragments" of $\mathcal{Q}(b, \mathcal{T})$ (or better, its theory) determine the initial segments of b. The following is the crucial lemma which is essentially due to Martin and Steel ([3])

Lemma 1.6 *Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ of limit length and s is a cofinal subset of $\delta(\mathcal{T})$; then there is at most one cofinal branch b such that there is $\alpha \in b$ with the property that $\pi^{\mathcal{T}}_{\alpha,b}$ exists and $s \subseteq rng(\pi^{\mathcal{T}}_{\alpha,b})$.*

Proof. Towards a contradiction, suppose there are two cofinal branches b and c such that for some α, β, both $\pi^{\mathcal{T}}_{\alpha,b}$ and $\pi^{\mathcal{T}}_{\beta,c}$ exist and $s \subseteq ran(\pi^{\mathcal{T}}_{\alpha,b}) \cap ran(\pi^{\mathcal{T}}_{\beta,c})$. Without loss of generality we can assume that α and β are the least ordinals with this property, $\alpha \leq \beta$ and that b and c diverge at α or earlier, i.e., if γ is the least ordinal in $b \cap c$ then $\gamma \leq \alpha$. By [3], we can assume that $b = \langle \alpha_n : n < \omega \rangle$, $c = \langle \beta_n : n < \omega \rangle$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Let then ξ be the least ordinal in $ran(\pi^{\mathcal{T}}_{\alpha,b}) \cap ran(\pi^{\mathcal{T}}_{\beta,c})$. Let n be the least such that $crit(\pi^{\mathcal{T}}_{\alpha_n,b}) > \xi$. This means that $crit(E^{\mathcal{T}}_{\alpha_{n+1}-1}) > \xi$ and that $lh(E^{\mathcal{T}}_{\alpha_n}) < \xi$. By the proof of Theorem 2.2 of [3], this means that for some $m \geq 1$, $\xi \in [crit(E^{\mathcal{T}}_{\beta_m -1}), lh(E^{\mathcal{T}}_{\beta_m -1}))$. This then implies that $\xi \notin ran(\pi^{\mathcal{T}}_{\beta_{m-1},c})$, which is a contradiction. □

$\mathcal{Q}(b, \mathcal{T})$ identifies b because it determines a canonical cofinal subset of $rng(i^{\mathcal{T}}_{\alpha,b} \cap \delta(\mathcal{T}))$, for some $\alpha \in b$, to which we can apply Lemma 1.6. But now, the proof of Lemma 1.6 gives the following refinement:

Lemma 1.7 *Suppose $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of limit length and b, c are two cofinal branches of $\mathcal{T}$ such that $\pi^{\mathcal{T}}_b$ and $\pi^{\mathcal{T}}_c$ exist. Suppose that for some α,*

$$\pi^{\mathcal{T}}_b(\alpha) = \pi^{\mathcal{T}}_c(\alpha) < \delta(\mathcal{T}).$$

Then $\pi^{\mathcal{T}}_b \restriction \alpha = \pi^{\mathcal{T}}_c \restriction \alpha$. Moreover, if $\xi \in b$ is the least such that $crit(i^{\mathcal{T}}_{\xi,b}) > \pi^{\mathcal{T}}_b(\alpha)$ then $\xi \in c$, so that $b \cap (\xi + 1) = c \cap (\xi + 1)$.

In addition to normal trees, we must consider linear stacks of normal trees. These are plays of the iteration game $G_k(\mathcal{M}, \alpha, \theta)$ in which II has not yet lost. See [17, Def. 4.4] for the formal definition. We shall generally use the vector notation $\vec{\mathcal{T}}$ for a stack of normal trees, and then $\mathcal{T}_\eta$ for its η^{th} normal component. $\mathcal{M}_\eta$ will denote the model at the beginning of the ηth round of $\vec{\mathcal{T}}$. If Σ is a (k, α, θ)-iteration strategy for $\mathcal{M}$, and $\vec{\mathcal{T}}$ is a stack of trees on $\mathcal{M}$ played according to Σ and having last model $\mathcal{N}$, then we let $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ be the (l, α, θ) strategy for $\mathcal{N}$ induced by Σ. (l is the degree of the branch $\mathcal{M}$-to-$\mathcal{N}$ of $\vec{\mathcal{T}}$. We assume here α is additively closed, so that there is such a strategy.) We say $\mathcal{M}$ is countably (α, θ)-iterable if all of its countable submodels are (α, θ)-iterable.

Suppose $\mathcal{M}$ is a mouse and Σ is an (α, θ)-iteration strategy. We then let

$I(\mathcal{M},\Sigma) = \{\mathcal{N} :$ there is a stack $\vec{\mathcal{T}}$ on $\mathcal{M}$ according to Σ with last model $\mathcal{N}$ and $\pi^{\vec{\mathcal{T}}}$ exists $\}$.

Given a premouse $\mathcal{M}$ with a unique Woodin cardinal δ, we let $\mathbb{B}^{\mathcal{M}}$ be the countably generated extender algebra of $\mathcal{M}$ at δ. In order to have a unique choice, we stipulate that the identities determining $\mathbb{B}^{\mathcal{M}}$ are precisely those coming from extenders E on the $\mathcal{M}$-sequence such that $\nu(E)$ is an inaccessible, but not a limit of inaccessibles, in $\mathcal{M}$, and $\nu(E) < \delta$. If $G \subseteq \mathbb{B}^M$ then we let x_G be the set naturally coded by G. For basic facts about the extender algebra, we refer the reader to [1] and [17].

2 AD$^+$ and the Solovay hierarchy

We will not need the exact formulation of AD^+. The interested readers can consult [19] or the introductory chapters of [6]. The *Solovay hierarchy* is a hierarchy of axioms extending AD^+. To define the hierarchy, we first need to define the *Solovay sequence*. First recall the *Wadge ordering* of $\mathcal{P}(\mathbb{R})$. For $A, B \subseteq \mathbb{R}$, we say A is *Wadge reducible* to B and write $A \leq_W B$ if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}[B] = A$. Martin showed that under AD, $\leq_W$ is a wellfounded relation. For $A \subseteq \mathbb{R}$, we let $w(A)$ be the rank of A in $\leq_W$. Under AD, $\Theta = \sup_{A\subseteq\mathbb{R}} w(A)$. The Solovay sequence is defined as follows.

Definition 2.1 (The Solovay sequence) *Assume AD. The Solovay sequence is a closed increasing sequence $\langle\theta_\alpha : \alpha \leq \Omega\rangle$ of ordinals defined by*

1. $\theta_0 = \sup\{\alpha :$ *there is an OD surjection* $f : \mathbb{R} \to \alpha\}$,
2. *if* $\theta_\beta < \Theta$ *then letting* $A \subseteq \mathbb{R}$ *be such that* $w(A) = \theta_\beta$,

$$\theta_{\beta+1} = \sup\{\alpha : \text{there is an } OD_A \text{ surjection } f : \mathcal{P}(\theta_\beta) \to \alpha\},$$

3. *if* λ *is a limit then* $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

The Solovay hierarchy is the hierarchy we get by requiring that Ω is large. The following are the first few theories of this hierarchy. $\leq_{con}$ is the consistency strength relation: given two theories T and S, $S \leq_{con} T$ if $Con(T) \vdash Con(S)$.

$$AD^+ + \Omega = 0 <_{con} AD^+ + \Omega = 1 < AD^+ + \Omega = 2 \cdot\cdot\cdot <_{con} AD^+ + \Omega = \omega <_{con} \cdots <_{con} \\ AD^+ + \Omega = \omega_1 <_{con} AD^+ + \Omega = \omega_1 + 1 \cdot\cdot\cdot.$$

For more on the Solovay hierarchy consult [5]. We will use the following theorem to construct a non-tame mouse from our hypothesis.

Theorem 2.2 (Steel-Woodin, [15]) *Assume there is a transitive inner model M containing $\mathbb{R}$ such that $M \vDash AD^+ + \Theta = \theta_1$. Then there is a non-tame mouse.*

3 Suitable mice

The core model induction is a method for constructing iteration strategies for various canonical structures. Often times the iteration strategies we construct are strategies for HOD of a certain model of determinacy. In many situations, including our current situation, HOD is a fine structural model. The proof of this fact will be used throughout this paper and that is why we take a moment to review the background material used in the analysis of HOD of models of determinacy. In later sections we will heavily rely on notions introduced in this section. In particular, notions such as *suitable premouse* or *quasi-iterable premouse* will be of crucial importance for us. Many of the notions introduced in this section are due to Woodin who introduced them in his computation of $\mathrm{HOD}^{L[x][g]}$ (see [18]). We start with suitability.

Often times, the notion of a suitable premouse is needed simultaneously in models of determinacy and in ZFC context. Our current situation is an instance of this and we chose to define the notion in a most general way to avoid confusions when applying it in different contexts.

Fix then some cardinal λ and let $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$. We assume $ZF + DC_\lambda$. Notice that any function $f : H_{\lambda^+} \to H_{\lambda^+}$ can be naturally coded by a subset of $\mathcal{P}(\lambda)$. We then let $Code_\lambda : H_{\lambda^+}^{H_{\lambda^+}} \to \mathcal{P}(\mathcal{P}(\lambda))$ be one such coding. If $\lambda = \omega$ then we just write $Code$. Because any λ^+-iteration strategy for a premouse of size $\leq \lambda$ is in $H_{\lambda^+}^{H_{\lambda^+}}$, we have that any such strategy is in the domain of $Code_\lambda$. Given a premouse $\mathcal{M}$, we say $\mathcal{M}$ *has an iteration strategy* in Γ if $|\mathcal{M}| \leq \lambda$ and $\mathcal{M}$ has a λ^+-iteration strategy (or (α, λ^+)-iteration strategy for $\alpha \leq \lambda^+$) Σ such that $Code_\lambda(\Sigma) \in \Gamma$. We let $Mice^\Gamma$ be the set of mice that have an iteration strategy in Γ. Given a countable set a we let

$$\mathcal{W}^\Gamma(a) = \cup\{\mathcal{N} : \mathcal{N} \text{ is a sound mouse over } a \text{ such that } \rho(\mathcal{N}) = a \text{ and } \mathcal{N} \in Mice^\Gamma\}$$

and define $Lp_\xi^\Gamma(a)$ for $\xi < \omega_1$ by induction as follows:

1. $Lp_0^\Gamma(a) = \mathcal{W}^\Gamma(a)$,

2. for $\xi < \omega_1$, $Lp^{\Gamma}_{\xi+1} = \mathcal{W}^{\Gamma}(Lp^{\Gamma}_{\xi}(a))$,

3. for limit $\xi < \omega_1$, $Lp^{\Gamma}_{\xi} = \cup_{\alpha<\xi} Lp^{\Gamma}_{\alpha}(a)$.

Definition 3.1 (Γ-suitable premouse) *A premouse $\mathcal{P}$ is Γ-suitable if there is a unique cardinal δ such that*

1. $\mathcal{P} \vDash$ "δ is the unique Woodin cardinal",

2. $o(\mathcal{P}) = \sup_{n<\omega}(\delta^{+n})^{\mathcal{P}}$,

3. for every $\mathcal{P}$-cardinal $\eta \neq \delta$, $\mathcal{W}^{\Gamma}(\mathcal{P}|\eta) \vDash$ "η isn't Woodin".

4. for any $\eta < o(\mathcal{P})$, $\mathcal{O}^{\mathcal{P}}_{\eta} = \mathcal{W}^{\Gamma}(\mathcal{P}|\eta)$.

Suppose $\mathcal{P}$ is Γ-suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 3.1. Given an iteration tree $\mathcal{T}$ on $\mathcal{P}$, we say $\mathcal{T}$ is *nice* if $\mathcal{T}$ has no fatal drops. Notice that Γ-suitable premice satisfy the hypothesis of Lemma 1.5. A nice tree $\mathcal{T}$ is Γ-*correctly guided* if for every limit $\alpha < lh(\mathcal{T})$, $\mathcal{Q}(\mathcal{T} \restriction \alpha)$ exists and

$$\mathcal{Q}(\mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{W}^{\Gamma}(\mathcal{M}(\mathcal{T} \restriction \alpha)).$$

$\mathcal{T}$ is Γ-*short* if it is nice, correctly guided and $\mathcal{W}^{\Gamma}(\mathcal{M}(\mathcal{T})) \vDash$ "$\delta(\mathcal{T})$ is not Woodin". $\mathcal{T}$ is Γ-*maximal* if it is nice, Γ-correctly guided yet not Γ-short. Notice that if $\mathcal{T}$ is a maximal tree and b is a branch such that $i^{\mathcal{T}}_b(\delta^{\mathcal{P}}) = \delta(\mathcal{T})$ then $\mathcal{T}$ doesn't have a nice normal continuation.

Definition 3.2 (Γ-correctly guided finite stack) *Suppose $\mathcal{P}$ is Γ-suitable. We say $\langle \mathcal{T}_i, \mathcal{P}_i : i < m \rangle$ is a Γ-correctly guided finite stack on $\mathcal{P}$ if*

1. $\mathcal{P}_0 = \mathcal{P}$,

2. $\mathcal{P}_i$ is Γ-suitable and $\mathcal{T}_i$ is a nice Γ-correctly guided tree on $\mathcal{P}_i$ below $\delta^{\mathcal{P}_i}$,

3. for every i such that $i+1 < m$ either $\mathcal{T}_i$ has a last model and $\pi^{\mathcal{T}_i}$-exists or $\mathcal{T}_i$ is maximal, and

(a) if $\mathcal{T}_i$ has a last model then $\mathcal{P}_{i+1}$ is the last model of $\mathcal{T}_i$ and if $lh(\mathcal{T}_i) = \alpha + 1$ where α is limit then if $\mathcal{T}_i^-$ is $\mathcal{T}_i$ without its last branch then $\mathcal{Q}(\mathcal{T}_i^-)$-exists and is an initial segment of $\mathcal{W}^{\Gamma}(\mathcal{M}(\mathcal{T} \restriction \alpha))$,

(b) *if* $\mathcal{T}_i$ *is* Γ*-maximal then* $\mathcal{P}_{i+1} = Lp^{\Gamma}_{\omega}(\mathcal{M}(\mathcal{T}_i))$.

Notice that if $\langle \mathcal{T}_i, \mathcal{P}_i : i \leq m\rangle$ is a correctly guided finite stack on $\mathcal{P}$ then only $\mathcal{T}_m$ can have a dropping last branch.

Definition 3.3 (The last model of a Γ-correctly guided finite stack) *Suppose* $\mathcal{P}$ *is* Γ*-suitable and* $\vec{\mathcal{T}} = \langle \mathcal{T}_i, \mathcal{P}_i : i \leq k\rangle$ *is a* Γ*-correctly guided finite stack on* $\mathcal{P}$. *We say* $\mathcal{R}$ *is the last model of* $\vec{\mathcal{T}}$ *if one of the following holds:*

1. $\mathcal{T}_k$ *has a last model and* $\mathcal{R}$ *is the last model of* $\mathcal{T}_k$,
2. $\mathcal{T}_k$ *is of limit length,* $\mathcal{T}_k$ *is* Γ*-short and there is a cofinal well-founded branch* b *such that* $\mathcal{Q}(b, \mathcal{T}_k)$ *exists,* $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}^{\Gamma}(\mathcal{M}(\mathcal{T}_k))$ *and* $\mathcal{R} = \mathcal{M}^{\mathcal{T}}_b$,
3. $\mathcal{T}_k$ *is of limit length,* $\mathcal{T}_k$ *is* Γ*-maximal,* $\mathcal{R}$ *is* Γ*-suitable and*

$$\mathcal{R} = Lp^{\Gamma}_{\omega}(\mathcal{M}(\mathcal{T}_k)).$$

We say $\mathcal{R}$ *is a* Γ*-correct iterate of* $\mathcal{P}$ *if there is a* Γ*-correctly guided finite stack on* $\mathcal{P}$ *with last model* $\mathcal{R}$.

Definition 3.4 ($S(\Gamma)$ and $F(\Gamma)$) *We let* $S(\Gamma) = \{\mathcal{Q} : \mathcal{Q}$ *is* Γ*-suitable*$\}$. *Also, we let* $F(\Gamma)$ *be the set of functions* f *such that* $dom(f) = S(\Gamma)$ *and for each* $\mathcal{P} \in S(\Gamma)$, $f(\mathcal{P}) \subseteq \mathcal{P}$ *and* $f(\mathcal{P})$ *is amenable to* $\mathcal{P}$, *i.e., for every* $X \in \mathcal{P}$, $X \cap f(\mathcal{P}) \in \mathcal{P}$.

Given $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$ we let $f^n(\mathcal{P}) = f(\mathcal{P}) \cap \mathcal{P}|((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}}$. Then $f(\mathcal{P}) = \cup_{n<\omega} f^n(\mathcal{P})$. We also let

$$\gamma^{\mathcal{P}}_f = \sup(\delta^{\mathcal{P}} \cap Hull^{\mathcal{P}}_1(\{f^n(\mathcal{P}) : n < \omega\})).$$

Notice that

$$\gamma^{\mathcal{P}}_f = \sup(\delta^{\mathcal{P}} \cap Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_f \cup \{f^n(\mathcal{P}) : n < \omega\})).$$

We then let

$$H^{\mathcal{P}}_f = Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_f \cup \{f^n(\mathcal{P}) : n < \omega\}).$$

If $\mathcal{P} \in S(\Gamma)$, $f \in F(\Gamma)$ and $i : \mathcal{P} \to \mathcal{Q}$ is an embedding then we let $i(f(\mathcal{P})) = \cup_{n<\omega} i(f^n(\mathcal{P}))$.

Definition 3.5 (f-iterability) *Suppose $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. We say $\mathcal{P}$ is f-iterable if whenever $\langle \mathcal{T}_k, \mathcal{P}_k : k < m\rangle$ is a finite correctly guided stack on $\mathcal{P}$ with last model $\mathcal{R}$ then there is a sequence $\langle b_k : k < m\rangle$ such that the following holds.*

1. *For $k < m-1$,*

$$b_k = \begin{cases} \emptyset & : \mathcal{T}_k \text{ has a successor length} \\ \text{cofinal well-founded branch} & \\ \text{such that } \mathcal{M}^{\mathcal{T}}_{b_k} = \mathcal{P}_\kappa & : \mathcal{T}_k \text{ is } \Gamma\text{-maximal} \end{cases}$$

2. *The following three cases hold.*
   (a) *If $\mathcal{T}_{m-1}$ has a successor length then $b_{m-1} = \emptyset$.*
   (b) *If $\mathcal{T}_{m-1}$ is Γ-short then there is a cofinal well-founded branch b such that $\mathcal{Q}(b, \mathcal{T}_{m-1})$ exists, $\mathcal{Q}(b, \mathcal{T}_{m-1}) \trianglelefteq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T}_{m-1}))$ and b_{m-1} is the unique such branch.*
   (c) *If $\mathcal{T}_{m-1}$ is Γ-maximal then b_{m-1} is a cofinal well-founded branch.*

3. *Letting*

$$\pi_k = \begin{cases} \pi^{\mathcal{T}_k} & : \mathcal{T}_k \text{ has a successor length} \\ \pi^{\mathcal{T}_k}_{b_k} & : \mathcal{T}_k \text{ is } \Gamma\text{-maximal} \end{cases}$$

   and $\pi = \pi_{m-1} \circ \pi_{m-2} \circ \cdots \pi_0$ then

$$\pi(f(\mathcal{P})) = f(\mathcal{R}).$$

Suppose again that $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. Suppose $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m\rangle$ is a Γ-correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{R}$. We say $\vec{b} = \langle b_k : k < m\rangle$ *witnesses f-iterability for* $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m\rangle$ if 2 above is satisfied. We then let

$$\pi_{\vec{\mathcal{T}},\vec{b},k} = \begin{cases} \pi^{\mathcal{T}_k} & : \mathcal{T}_k \text{ has a successor length} \\ \pi^{\mathcal{T}_k}_{b_k} & : \mathcal{T}_k \text{ is maximal} \\ \text{undefined} & : \text{otherwise} \end{cases}$$

and $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi_{\vec{\mathcal{T}},\vec{b},m-1} \circ \pi_{\vec{\mathcal{T}},\vec{b},m-2} \circ \cdots \pi_{\vec{\mathcal{T}},\vec{b},0}$. Notice that clause three isn't vacuous as it might be that $\mathcal{T}_k$ is Γ-short and its unique branch has a drop.

Continuing with the notation of the previous paragraph, let $\vec{b}$ and $\vec{c}$ be two f-iterability branches for $\vec{\mathcal{T}}$. It then follows from Theorem 1.6 that

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H_f^{\mathcal{P}}.$$

Lemma 3.6 (Uniqueness of f-iterability embeddings) *Suppose $\mathcal{P} \in S(\Gamma)$, $f \in F(\Gamma)$ and $\vec{\mathcal{T}}$ is a finite correctly guided stack on $\mathcal{P}$. Suppose $\vec{b}$ and $\vec{c}$ are two f-iterability branches for $\vec{\mathcal{T}}$. Then*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H_f^{\mathcal{P}}.$$

Moreover, if $\vec{\mathcal{T}}$ consists of just one normal tree $\mathcal{T}$, $\mathcal{Q}$ is the last model of $\mathcal{T}$ and b and c witness f-iterability for $\mathcal{T}$ then if $\xi \in b$ is the least such that $crit(E_\xi^{\mathcal{T}}) > \gamma_f^{\mathcal{Q}}$ then $b \cap \xi = c \cap \xi$.

Definition 3.7 *Suppose $\mathcal{P} \in S(\Gamma)$ and f-iterable. Given a Γ-correctly guided maximal $\mathcal{T}$ on $\mathcal{P}$ with last model $\mathcal{Q}$, we let $b_{\mathcal{T},f} = b \cap \xi$ where b witnesses f-iterability of $\mathcal{P}$ for $\mathcal{T}$ and $\xi \in b$ is the least such that $crit(\pi_{\xi,b}^{\mathcal{T}}) > \gamma_f^{\mathcal{Q}}$.*

When $\mathcal{T}$ is clear from context, we will omit it from our notation and just write b_f. Notice that if $\mathcal{P}$ is f-iterable, $\vec{\mathcal{T}}$ is a Γ-correctly guided finite stack on $\mathcal{P}$, and $\vec{b}$ witnesses f-iterability of $\mathcal{P}$ for $\vec{\mathcal{T}}$ then even though $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}}$ is independent of $\vec{b}$ it may depend on $\vec{\mathcal{T}}$. This observation motivates the following definition.

Definition 3.8 (Strong f-iterability) *Suppose $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. We say $\mathcal{P}$ is* strongly f-iterable *if $\mathcal{P}$ is f-iterable and whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_j, \mathcal{P}_j : j < u \rangle \in H_{\lambda^+}$ and $\vec{\mathcal{U}} = \langle \mathcal{U}_j, \mathcal{Q}_j : j < v \rangle \in H_{\lambda^+}$ are two correctly guided finite stacks on $\mathcal{P}$ with common last model $\mathcal{R}$, $\vec{b}$ witnesses f-iterability for $\vec{\mathcal{T}}$ and $\vec{c}$ witnesses f-iterability for $\vec{\mathcal{U}}$ then $\pi_b^{\vec{\mathcal{T}}}$ is defined iff $\pi_c^{\vec{\mathcal{U}}}$ is defined and*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{c}} \restriction H_f^{\mathcal{P}}.$$

If $\mathcal{P}$ is strongly f-iterable and $\vec{\mathcal{T}}$ is a Γ-correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{R}$ then we let

$$\pi_{\mathcal{P},\mathcal{R},f} : H_f^{\mathcal{P}} \to H_f^{\mathcal{R}}$$

be the embedding given by any $\vec{b}$ which witnesses the f-iterability of $\vec{\mathcal{T}}$, i.e., fixing $\vec{b}$ which witnesses f-iterability for $\vec{\mathcal{T}}$,

$$\pi_{\mathcal{P},\mathcal{R},f} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}}.$$

Clearly, $\pi_{\mathcal{P},\mathcal{R},f}$ is independent of $\vec{\mathcal{T}}$ and $\vec{b}$.

Given a finite sequence of functions $\vec{f} = \langle f_i : i < n\rangle \in F(\Gamma)$, we let $\oplus_{i<n} f_i \in F(\Gamma)$ be the function given by $(\oplus_{i<n} f_i)(\mathcal{P}) = \langle f_i(\mathcal{P}) : i < n\rangle$. We set $\oplus\vec{f} = \oplus_{i<n} f_i$.

Given $F \subseteq F(\Gamma)$, we let

$$\mathcal{I}_{\Gamma,F} = \{(\mathcal{P}, \vec{f}) : \mathcal{P} \in S(\Gamma),\ \vec{f} \in (F)^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } \oplus\vec{f}\text{-iterable}\}.$$

Definition 3.9 *Given $F \subseteq F(\Gamma)$, we say F is closed if for any $\vec{f} \in F^{<\omega}$ there is $\mathcal{P}$ such that $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{\Gamma,F}$ and for any $\vec{g} \in F^{<\omega}$, there is a Γ-correct iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q}, \vec{f}\cup\vec{g}) \in \mathcal{I}_{\Gamma,F}$.*

Fix now a closed $F \subseteq F(\Gamma)$. Let

$$\mathcal{F}_{\Gamma,F} = \{H^{\mathcal{P}}_f : (\mathcal{P}, f) \in \mathcal{I}_{\Gamma,F}\}.$$

We then define $\preceq_{\Gamma,F}$ on $\mathcal{I}_{\Gamma,F}$ by letting $(\mathcal{P}, \vec{f}) \preceq_{\Gamma,F} (\mathcal{Q}, \vec{g})$ iff $\mathcal{Q}$ is a Γ-correct iterate of $\mathcal{P}$ and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{P}, \vec{f}) \preceq_{\Gamma,F} (\mathcal{Q}, \vec{g})$, we have that

$$\pi_{\mathcal{P},\mathcal{Q},\vec{f}} : H^{\mathcal{P}}_{\oplus\vec{f}} \to H^{\mathcal{Q}}_{\oplus\vec{g}}.$$

Notice that if F is closed then $\preceq_{\Gamma,F}$ is directed. Let then

$$\mathcal{M}_{\infty,\Gamma,F}$$

be the direct limit of $(\mathcal{F}_{\Gamma,F}, \preceq_{\Gamma,F})$ under $\pi_{\mathcal{P},\mathcal{Q},\vec{f}}$. Given $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{\Gamma,F}$, we let $\pi_{\mathcal{P},\vec{f},\infty} : H^{\mathcal{P}}_{\oplus\vec{f}} \to \mathcal{M}_{\infty,\Gamma,F}$ be the direct limit embedding.

Lemma 3.10 *$\mathcal{M}_{\infty,\Gamma,F}$ is wellfounded.*

Proof. If not then we can fix $\langle(\mathcal{P}_i, f_i) : i < \omega\rangle \subseteq F$ and $\langle\alpha_i : i < \omega\rangle$ such that $\alpha_i \in H^{\mathcal{P}_i}_{f_i}$, $(\mathcal{P}_i, \oplus_{j\leq i} f_j) \in \mathcal{I}_{\Gamma,F}$ and $\pi_{\mathcal{P}_{i+1},f_{i+1},\infty}(\alpha_{i+1}) < \pi_{\mathcal{P}_i,f_i,\infty}(\alpha_i)$. By simultaneously comparing $\mathcal{P}_i$'s we get a common Γ-correct iterate $\mathcal{P}$ such that if $\beta_i = \pi_{\mathcal{P}_i,\mathcal{P},f_i}(\alpha_i)$ then $\beta_{i+1} < \beta_i$, contradiction!
□

It turns out that $V^{\mathrm{HOD}}_{\Theta}$ of many models of determinacy can be obtained as $\mathcal{M}_{\infty,\Gamma,F}$ for some Γ and F. We will give the details in the next few subsection.

4 How to read this paper

While we have made an attempt to make the paper largely self-contained, we believe that some experience with the basic notions of inner model theory and descriptive set theory will be very useful. In this section, we attempt to explain how to read the paper and what points to concentrate on.

The main tool used in this paper is the core model induction, which is a method originally due to Woodin. It can be viewed either as a method for inductively constructing more and more complicated mice or as a method for inductively constructing more and more complicated models of determinacy. It is the author's belief that the second is a more useful point of view. This is because while it is conjectured and expected to be true that complicated models of determinacy have complicated mice, it is difficult and perhaps even impossible to capture the essence of this claim via just pure inner model theoretic notions without using some descriptive set theoretic notion of complexity such as Wadge order. The best one can do is to state the *Mouse Set Conjecture*, which is the conjecture referred to in the previous sentence.

The Mouse Set Conjecture, MSC. Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then for $x, y \in \mathbb{R}$, $x \in OD(y)$ if and only if there is a sound ω_1-iterable y-mouse $\mathcal{M}$ such that $x \in \mathcal{M}$.

MSC is one of the most central open problems in descriptive inner model theory. We refer the reader to [5] for more information on its role in set theory. We let *Mouse Capturing* (MC) stand for the conclusion of the MSC. The author showed in [6] that

Theorem 4.1 ([6]) *MC holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular".*

MC is crucial in core model induction constructions. It is what one usually uses to show that the new set just constructed is more complicated than all the previous ones. The reader will encounter this sort of argument several times throughout the paper. For instance, see the proof of Lemma 9.35 or the conclusion of Subsection 9.8. We give an outline of how this conclusion is made in the following remark.

Remark 4.2 *The argument we are about to given is by now a standard argument. It is based on Woodin's proof that $\mathcal{M}_1 \vDash$ "V is not iterable", where $\mathcal{M}_1$ is the minimal proper class model with a Woodin cardinal.*

Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is some pointclass and $(\mathcal{P}, \Sigma)$ is a pair such that $\mathcal{P}$ is Γ-suitable premouse and Σ is Γ-fullness preserving iteration strategy. Suppose further that $L(\Gamma, \mathbb{R}) \vDash AD^+ + MC + \theta_0 = \Theta$. Then $\Sigma \notin \Gamma$.

Assuming $\Sigma \in \Gamma$, one gets a contradiction as follows. Work in $L(\Gamma, \mathbb{R})$. Since we are assuming $\theta_0 = \Theta$, there is an x such that Σ is ordinal definable from x. It then follows from MC that for all y Turing above x, $\Sigma \cap Lp(y) \in Lp(y)$ (Lp is defined at the beginning of Section 9). Iterate $\mathcal{P}$ to get $\mathcal{Q}$ that makes x generic over the extender algebra of $\mathcal{Q}$. We then have that if Λ is the strategy of $\mathcal{Q}$ induced by Σ then

$$\Lambda \cap \mathcal{Q}[x] \in \mathcal{Q}[x]$$

However, this cannot happen. To see this, consider the maximal tree $\mathcal{T}$ on $\mathcal{Q}|((\delta^{\mathcal{Q}})^+)^{\mathcal{Q}}$ according to Λ that makes $\mathcal{Q}|((\delta^{\mathcal{Q}})^+)^{\mathcal{Q}}$ generic. Suitability of $\mathcal{Q}$ and fullness preservation of Λ implies that $\mathcal{T} \in \mathcal{Q}$ and $\delta(\mathcal{T}) = ((\delta^{\mathcal{Q}})^{++})^{\mathcal{Q}}$. The final branch of $\mathcal{T}$ given by Λ, however, cannot be in $\mathcal{Q}[x]$ as it singularizes $((\delta^{\mathcal{Q}})^{++})^{\mathcal{Q}}$.

All core model induction applications are done by making a minimality assumption. In this paper, our minimality assumption is the following. *Good* is defined in Definition 9.2.

Minimality Assumption. For any good μ and a generic $g \subseteq Coll(\omega, \mu)$, in $V[g]$, there is no inner model M such that $Ord, \mathbb{R} \subseteq M$ and

$$M \vDash AD^+ + \Theta = \theta_1.$$

We proceed by assuming our Minimality Assumption and derive a contradiction. It follows from Theorem 2.2 that the existence of an M as in the Minimality Assumptions implies the existence of a non-tame mouse. This then completes the proof of Theorem 0.1.

While doing core model induction one encounters two steps that can be characterized as *internal* and *external*. In the internal step of the induction, there is a concrete model M and our goal is to show that $M \vDash AD^+$. M is usually a fine structural model of $\mathbb{R}$. In the current situation, we do not do the internal step of the induction as this was done by John Steel in [14]. Theorem 9.3 summarizes Steel's work. In that theorem, $L^{\Sigma}(\mathbb{R})$ is the universe M mentioned above. Another place where such an internal induction is taking place is Theorem 9.6. At various points in the paper, we conclude that certain models satisfy AD^+. All of this is done via a core model induction such as the one done by Steel in [14]. The following, however, is an important remark.

Remark 4.3 *To prove that $\mathcal{S}_{\mu,g} \vDash AD^+$, where $\mathcal{S}_{\mu,g}$ is as in Theorem 9.6, we need to use [13] and [16] and the core model induction of [14]. The scales analysis of [13] and [16] is carried out for those levels $\mathcal{S}$ of $Lp(\mathbb{R})$ that satisfy $V = \mathrm{HOD}$ (Lp is defined at the begging of Section 9). However, this is all one needs. This is because any level $\mathcal{S}$ of $Lp(\mathbb{R})$ such that $\mathcal{S} \vDash V \neq \mathrm{HOD}$ is usually contained in another which satisfies $AD^+ + V = \mathrm{HOD}$. Thus, it is enough to do the induction over the levels of $Lp(\mathbb{R})$ that satisfy $V = \mathrm{HOD}$.*

To see this, fix a level $\mathcal{S}$ of $Lp(\mathbb{R})$ such that $\mathcal{S}$ ends a gap and $\mathcal{S} \vDash V \neq \mathrm{HOD}$. It then follows that some countable substructure of $\mathcal{S}$ doesn't have an iteration strategy in $\mathcal{S}$. Fix such a $\pi : \mathcal{M} \to \mathcal{S}$. Let Σ be the strategy of $\mathcal{M}$. We can use Theorem 9.3, Theorem 9.9 and core model induction to conclude that $L^{\Sigma}(\mathbb{R}) \vDash AD^+$. Because Σ is just a strategy for a mouse, it follows that if $M = L(\Sigma, \mathbb{R})$ then $M \vDash \theta_0 = \Theta$. It is shown in [7] that such models have the form $M = L((Lp(\mathbb{R}))^M)$. It then immediately follows that $\mathcal{S} \trianglelefteq (Lp(\mathbb{R}))^M \trianglelefteq Lp(\mathbb{R})$ (because Σ is in second but not first). We clearly have that $(Lp(\mathbb{R}))^M \vDash AD^+ + V = \mathrm{HOD}$.

In this paper, we are mostly concerned with the external core model induction. In our case, the goal is to construct an iteration strategy Σ which is not in $M = L(Lp(\mathbb{R}))$ (because we are only outlining the argument, we do not give the precise definition of M, which is somewhat different). Our approach is the traditional approach dating back to Ketchersid's work (see [2]).

We let $\mathcal{P}$ be, essentially, HOD of M and find some $(\mathcal{Q}, \Sigma)$ such that $\mathcal{Q}$ is countable, $\mathcal{P}(\mathbb{R})^M$-suitable and Σ is a $\mathcal{P}(\mathbb{R})^M$-fullness preserving iteration strategy with branch condensation. Moreover, the direct limit of all iterates of $\mathcal{Q}$ is $\mathcal{P}$. It then easily follows that $\Sigma \notin M$ and (via a use of core model induction) $L^{\Sigma}(\mathbb{R}) \vDash AD^+$. We then conclude as in Remark 4.2 that $L(\Sigma, \mathbb{R}) \vDash \theta_0 < \Theta$.

$(\mathcal{Q}, \Sigma)$ is constructed by taking Skolem hulls of $\mathcal{P}$. Failure of $\square_\kappa$ is used to conclude that one can find *full* hulls of $\mathcal{P}$ (see, for instance, Lemma 9.8, Lemma 9.11 and Theorem 9.13). $\mathcal{Q}$ is one such hull. Letting $\pi : \mathcal{Q} \to \mathcal{P}$ be the uncollapse map, Σ is constructed using the direct limit construction outlined in Section 8 using Theorem 8.3 and ideas from Section 6. Σ has the property that whenever $\mathcal{S}$ is a Σ-iterate of $\mathcal{Q}$ and $\tau : \mathcal{Q} \to \mathcal{S}$ is the iteration embedding then there is $\sigma : \mathcal{S} \to \mathcal{P}$ such that $\pi = \sigma \circ \tau$. This property guarantees fullness preservation of Σ (see Corollary 9.22).

The hardest part is to show that Σ has branch condensation. This is the most important distinction between our current situation and [2]. In the later, assuming Σ doesn't have branch condensation, the existence of various ideals on ω_1 allows one to obtain a "bad" sequence of length ω_1 inside a determinacy model. One can then show that such bad sequence of length ω_1 cannot exist under AD^+.

In our current case, we do not have such an ideal. The solution is to implement the ideas from Section 7. To use the material of Section 7, we need an extension of $\mathcal{Q}$, $\mathcal{S}$, with ω many Woodin cardinals. We also need to extend Σ to a strategy for $\mathcal{S}$. Call this new strategy Λ. All of what we have just said is done in Subsection 9.4 and Subsection 9.5. The rest of the argument is essential as follows.

The argument that we are about to outline resembles the argument that some tail of the iteration strategy of $\mathcal{M}_\omega$ strongly respects a given $OD^{L(\mathbb{R})}$ set of reals. Fix an $A \in \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$. We would like to find a Λ-iterate $\mathcal{W}$ of $\mathcal{S}$ such that $\Lambda_{\mathcal{W}}$, the strategy of $\mathcal{W}$ induced by Λ, strongly respects A. If we do this for enough A, then we can show that some tail of Λ has branch condensation as in Section 6 (in particular, see Lemma 6.6). Thus, it is enough to show how to do this for one A.

What helps us is the fact that we can realize $Lp(\mathbb{R})$ as a derived model of an iterate of $\mathcal{S}$. This is done in Subsection 9.7. If now our chosen A is bad, we can internalize it to this iterate of $\mathcal{S}$ and show that we cannot have an ω-sequence of iterations witnessing that A isn't strongly respected.

5 Quasi-iterability

We are now in a position to define quasi-iterability which will later be used to construct strategies for certain HODs of models of determinacy. We continue with the set up of Section 3. Thus, recall that we are assuming ZF and $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$ for some cardinal λ. Fix some Γ-suitable $\mathcal{P}$ and a closed $F \subseteq F(\Gamma)$. Let $G \subseteq F$.

Definition 5.1 *We say $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G)-quasi iteration of $\mathcal{P}$ of length ν if*

1. *$\mathcal{M}_0 = \mathcal{P}$ and for all $\alpha < \nu$, $\mathcal{M}_\alpha$ is Γ-suitable,*
2. *if $\alpha < \nu$ then $\mathcal{T}_\alpha$ is a Γ-correctly guided tree on $\mathcal{M}_\alpha$, and if $\alpha + 1 < \nu$ then $\mathcal{M}_{\alpha+1}$ is the last model of $\mathcal{T}_\alpha$,*
3. *for all $\alpha < \nu$, $\mathcal{M}_\alpha$ is strongly f-iterable for every $f \in G$,*

4. for all limit $\alpha < \nu$, for any $g \in F$ there is $\beta < \alpha$ such that for any $\gamma \in [\beta, \alpha)$

(a) if $lh(\mathcal{T}_\gamma)$ is a successor then $\pi^{\mathcal{T}_\gamma}(g(\mathcal{M}_\gamma)) = g(\mathcal{M}_{\gamma+1})$,

(b) if $lh(\mathcal{T}_\gamma)$ is limit then there is a cofinal wellfounded branch b of $\mathcal{T}_\gamma$ such that $\mathcal{M}_b^{\mathcal{T}_\gamma} = \mathcal{M}_{\gamma+1}$ and $\pi_b^{\mathcal{T}_\gamma}(g(\mathcal{M}_\gamma)) = g(\mathcal{M}_{\gamma+1})$,

Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G)-quasi iteration of $\mathcal{P}$ of length ν. Given limit $\alpha < \nu$ and $g \in F$ we let $\beta_{g,\alpha} < \alpha$ be the least ordinal satisfying 4 of Definition 5.1.

Definition 5.2 *Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G)-quasi iteration of $\mathcal{P}$ of length ν. We say $\langle \pi^{g,\alpha}_{\gamma,\xi} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in Lim \cap \nu+1 \rangle$ are the embeddings of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ if for all limit $\alpha \leq \nu$ and for all $g \in F$*

1. $\pi^{g,\alpha}_{\gamma,\xi}$ is defined for all $\gamma, \xi \in [\beta_{g,\alpha}, \alpha]$,

2. $\pi^{g,\alpha}_{\gamma,\gamma+1} = \pi^{\mathcal{T}_\gamma}$ if it exists and otherwise $\pi^{g,\alpha}_{\gamma,\gamma+1} = \pi_b^{\mathcal{T}_\gamma} \restriction H_g^{\mathcal{M}_\gamma}$ where b witnesses clause 4b for α and g,

3. fixing $\gamma \in [\beta_{g,\alpha}, \alpha]$, $\pi^{g,\alpha}_{\gamma,\xi}$ is defined by induction on $\xi \in (\gamma, \alpha]$ where the first step of the induction is clause 2 above and the rest is given by the following scheme:

(a) if $\xi \leq \alpha$ is limit then if $H_g^{\mathcal{M}_\xi}$ is the direct limit of $H_g^{\mathcal{M}_\lambda}$ under $\pi^{g,\alpha}_{\gamma,\lambda}$ for $\gamma < \lambda < \xi$ then $\pi^{g,\alpha}_{\gamma,\xi}$ is the direct limit embedding and otherwise it is undefined,

(b) if $\xi = \lambda + 1$ then $\pi^{g,\alpha}_{\gamma,\xi} = \pi^{g,\alpha}_{\lambda,\lambda+1} \circ \pi^{g,\alpha}_{\gamma,\lambda}$.

Next we define (F, G)-quasi iterations.

Definition 5.3 *Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G)-quasi iteration of $\mathcal{P}$ of length ν. We say $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a (F, G)-quasi iteration of $\mathcal{P}$ of length ν if letting $\langle \pi^{g,\alpha}_{\gamma,\xi} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in Lim \cap \nu + 1 \rangle$ be the embeddings of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ then*

1. for every α, $\mathcal{M}_\alpha = \cup_{g \in F} H_g^{\mathcal{M}_\alpha}$, and

2. for every limit $\alpha < \nu$ and for every $g \in F$, $H_g^{\mathcal{M}_\alpha}$ is the direct limit of $H_g^{\mathcal{M}_\gamma}$ for $\gamma \in [b_{g,\alpha}, \alpha)$ under the maps $\pi^{g,\alpha}_{\gamma,\xi}$ for $\gamma < \xi \in [\beta_{g,\alpha}, \alpha)$.

Finally we define the last model of (F, G)-quasi iterations.

Definition 5.4 *Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+\rangle$ is a (F,G)-quasi iteration of $\mathcal{P}$ of length ν with embeddings $\langle \pi^{g,\alpha}_{\gamma,\xi} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in Lim \cap \nu+1\rangle$. Then we say $\mathcal{Q}$ is the last model of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+\rangle$ if $\mathcal{Q}$ is Γ-suitable, $\mathcal{Q}$ is strongly f-iterable for all $f \in G$ and one of the following holds:*

1. *$\nu = \alpha+1$, $\mathcal{Q}$ is the last model of $\mathcal{T}_\alpha$ as defined in Definition 3.3 and $\mathcal{Q} = \cup_{f\in F} H^{\mathcal{Q}}_f$ or*

2. *ν is limit, $\mathcal{Q} = \cup_{g\in F} H^{\mathcal{Q}}_g$ and for $g \in F$, $H^{\mathcal{Q}}_g$ is the direct limit of $H^{\mathcal{M}_\gamma}_g$ under $\pi^{g,\nu}_{\gamma,\xi}$ for $\gamma < \xi \in [\beta_{g,\nu}, \nu)$.*

Notice that it may be the case that quasi iterations don't have last models but when they do it is uniquely determined.

Definition 5.5 *We say $\mathcal{P}$ is (F,G)-quasi iterable if all of its (F,G)-quasi iterations have a last model.*

Before moving on we fix some notation. When $G = \{f\}$ then we use (F,f) instead of (F,G). When $G = \emptyset$ then we write F instead of $(F,\emptyset)$. Suppose $F \subseteq F(\Gamma)$ is closed. We say $\mathcal{P}$ is (strongly) F-iterable if $\mathcal{P}$ is (strongly) f-iterable for all $f \in F$. Suppose now that $\mathcal{P}$ is (strongly) F-iterable. Let $\mathcal{Q}$ be a Γ-correctly guided iterate of $\mathcal{P}$. Then we let $\pi_{\mathcal{P},\mathcal{Q},F} = \cup_{f\in F}\pi_{\mathcal{P},\mathcal{Q},f}$. Suppose further there is some $F^* \subseteq F(\Gamma)$ such that $\mathcal{P}$ is (F^*,F)-quasi iterable and suppose that $\mathcal{Q}$ is a (F^*,F)-quasi iterate of $\mathcal{P}$. We then let $\pi_{\mathcal{P},\mathcal{Q},F} : \cup_{f\in F} H^{\mathcal{P}}_f \to \cup_{f\in F} H^{\mathcal{Q}}_f$ be the iteration embedding coming from the composition of the quasi iterability embeddings. We also define $\pi_{\mathcal{Q},F,\infty}$ similarly.

6 $\vec{f}$-guided strategies

We continue with the set up of Section 3. Thus, recall that we are assuming $ZF + DC_\lambda$. In this subsection, our goal is to develop tools for constructing strategies by putting together pieces of various f-iterability branches. In later sections, we will need to construct iteration strategies that have the so-called branch condensation and one method for producing such strategies is via producing strongly $\vec{f}$-guided strategies. Below we let ZFC^- stand for ZFC without Replacement or without Powerset Axiom. We start by introducing *branch condensation*, a notion that played an important role in [6].

Definition 6.1 *Suppose Σ is an (α, β)-iteration strategy for some structure M such that $M \vDash ZFC^-$ (M need not be a fine structural model). We say Σ has branch condensation if whenever $(\mathcal{R}, \vec{\mathcal{T}}, \mathcal{Q}, \vec{\mathcal{U}}, c, \pi)$ is such that*

1. *$\vec{\mathcal{T}}$ is a stack on M according to Σ with last model $\mathcal{R}$ and $\pi^{\vec{\mathcal{T}}}$ exists,*
2. *$\vec{\mathcal{U}}$ is a stack on M according to Σ such that its last normal component is of limit length and c is some branch on $\vec{\mathcal{U}}$ such that $\pi_c^{\vec{\mathcal{U}}}$ exists,*
3. *$\mathcal{Q} = \mathcal{M}_c^{\vec{\mathcal{U}}}$ and $\pi : \mathcal{Q} \to_{\Sigma_1} \mathcal{R}$ is such that $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}$*

then $c = \Sigma(\vec{\mathcal{U}})$.

In all of our applications, the embedding π in clause 3 will actually be fully elementary. Below we introduce one of the main methods for producing strategies with branch condensation. The basic idea is that if the strategy moves many operators correctly then it must have branch condensation. For example, let $\mathcal{M} = \mathcal{M}_1|(\delta^+)^{\mathcal{M}_1}$ where $\mathcal{M}_1$ is the minimal mouse with a Woodin cardinal and δ is the Woodin cardinal of $\mathcal{M}_1$. Then $\mathcal{M}$ may have many iteration strategies but it has one strategy that moves the theory of sharps correctly. Let Σ be this strategy. It can be shown, using the fact that the theory of sharps *condenses*, that Σ has branch condensation. Below we will explain how exactly the aforementioned condensation of the theory of sharps works in a more general context.

Definition 6.2 *Suppose $\mathcal{P} \in S(\Gamma)$ and Σ is a (λ^+, λ^+)-iteration strategy for $\mathcal{P}$. We say Σ is Γ-fullness preserving if whenever $i : \mathcal{P} \to \mathcal{Q}$ comes from an iteration produced by Σ, $\mathcal{Q} \in S(\Gamma)$. Given $f \in F(\Gamma)$, we say Σ respects f if whenever $i : \mathcal{P} \to \mathcal{Q}$ and $j : \mathcal{Q} \to \mathcal{R}$ are iterations produced via Σ then $j(f(\mathcal{Q})) = f(\mathcal{R})$. We say Σ strongly respects f if whenever $i, j, \mathcal{P}, \mathcal{Q}, \mathcal{R}$ are as above and $\mathcal{S}$ is such that there are $\sigma : \mathcal{Q} \to \mathcal{S}$ and $\tau : \mathcal{S} \to \mathcal{R}$ such that $j = \tau \circ \sigma$ then $\mathcal{S} \in S(\Gamma)$ and $\sigma(f(\mathcal{Q})) = f(\mathcal{S})$.*

If Σ strongly respects many functions f then it is possible to show that Σ has branch condensation. Fix some $F \subseteq F(\Gamma)$.

Definition 6.3 *We say F is a quasi-self-justifying-system (qsjs) if there is $\mathcal{P} \in S(\Gamma)$ such that*

1. *F is closed,*

2. *for any $\vec{f} \in F^{<\omega}$, $\oplus\vec{f} \in F$,*

3. *for every $f \in F$, $\mathcal{P}$ is (F, f)-quasi iterable,*

4. *for any $f \in F$, whenever $\mathcal{Q}$ is an (F, f)-quasi iterate of $\mathcal{P}$, $\sup_{g\in F} \gamma_g^{\mathcal{Q}} = \delta^{\mathcal{Q}}$ (this condition follows from clause 3 above),*

5. *whenever $\mathcal{Q}$ is an F-quasi iterate of $\mathcal{P}$ and $\sigma : \mathcal{R} \to_{\Sigma_1} \mathcal{Q}$ is such that $f(\mathcal{Q}) \in rng(\sigma)$ for all $f \in F$ then $\mathcal{R} \in S(\Gamma)$.*

The following lemmas are essentially due to Woodin and are based on Lemma 1.6. They collectively imply that if F is a qsjs then it produces a Γ-fullness preserving iteration strategy for some $\mathcal{P} \in S(\Gamma)$.

Lemma 6.4 (ZF) *Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then there is a λ^+-iteration strategy Λ^F for $\mathcal{P}$ that is Γ-fullness preserving.*

Proof. We define Λ^F for normal trees. First we describe Λ^F on Γ-correctly guided trees. Given such a tree $\mathcal{T}$ on $\mathcal{P}$ we have two cases. If $\mathcal{T}$ is Γ-short then we let $\Lambda^F(\mathcal{T}) = b$ be the unique branch of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$-exists. If $\mathcal{T}$ is a maximal then we do the following. Let $\mathcal{Q}$ be the last model of $\mathcal{T}$. Recall the definition of b_f for $f \in F$ (see Definition 3.7). Because $\sup_{f\in F} \gamma_f^{\mathcal{Q}} = \delta^{\mathcal{Q}}$, we have that $b = \cup_{f\in F} b_f$ is a cofinal branch of F. Notice now that if $\mathcal{S} = \cup_{f\in F} \pi_b^{\mathcal{T}}(H_f^{\mathcal{P}})$ then there is $\sigma : \mathcal{S} \to_{\Sigma_1} \mathcal{Q}$ and $\tau : \mathcal{P} \to \mathcal{S}$ such that $\pi_b^{\mathcal{T}} = \sigma \circ \tau$. By clause 5 of Definition 6.3, we have that $\mathcal{S} \in S(\Gamma)$ and by clause 4 of Definition 6.3, we have that $\sigma \restriction \delta(\mathcal{T}) = id$. Hence, $\mathcal{S} = \mathcal{Q}$. We then let $\Lambda^F(\mathcal{T}) = b$.

If $\mathcal{T}$ has a fatal drop then let (α, η) be such that $\mathcal{T}$ has a fatal drop at (α, η). Then we set $\Lambda^F(\mathcal{T}) = \Lambda(\mathcal{T}^*)$ where Λ is the strategy of $\mathcal{O}_\eta^M$ and $\mathcal{T}^*$ is $\mathcal{T}$ after stage α. □

Notice that if F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it then any F-quasi iterate of $\mathcal{P}$ also witnesses that F is a qsjs. Given then any $\mathcal{Q} \in S(\Gamma)$ witnessing that F is a qsjs we let $\Lambda_{\mathcal{Q}}^F$ be the λ^+-iteration strategy given by Lemma 6.4. Let then $\mathcal{P} \in S(\Gamma)$ witness that F is qsjs. To get a (λ^+, λ^+)-strategy for $\mathcal{P}$, we just need to dovetail all the strategies of the form $\Lambda_{\mathcal{Q}}^F$ where $\mathcal{Q}$ ranges over F-quasi iterates of $\mathcal{P}$. The following notion will be used in the proof of the next lemma.

Given an iteration tree $\mathcal{T}$ such that $lh(\mathcal{T}) = \gamma + 1$ where γ is limit, we let $\mathcal{T}^-$ be $\mathcal{T}$ without its last branch. Suppose $\vec{\mathcal{T}}$ is a stack on a Γ-suitable premouse $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is *good* if letting

$\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \eta \rangle$ be the normal components of $\vec{\mathcal{T}}$, for each $\alpha < \eta$, $\mathcal{M}_\alpha$ is Γ-suitable and $\mathcal{T}_\alpha$ is a Γ-correctly guided tree on $\mathcal{M}_\alpha$. We let $\vec{\mathcal{T}}^*$ be defined as follows: $\vec{\mathcal{T}}^* = \langle \mathcal{U}_\alpha, \mathcal{M}_\alpha : \alpha < \eta \rangle$ where $\mathcal{U}_\alpha = \mathcal{T}_\alpha$ if either $\mathcal{T}_\alpha^-$ is undefined or it is defined but it is Γ-short and otherwise, $\mathcal{U}_\alpha = \mathcal{T}_\alpha^-$. We say $\vec{\mathcal{T}}^*$ is the quasi-rearrangement of $\vec{\mathcal{T}}$.

Lemma 6.5 (ZF) *Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then there is a (λ^+, λ^+)-iteration strategy Σ^F for $\mathcal{P}$ such that Σ^F is Γ-fullness preserving.*

Proof. We handle trees with fatal drops exactly the same way as in Lemma 6.4. We leave the details to the reader. By induction we define $\langle \Sigma^F_\alpha : \alpha < \lambda^+ \rangle$ such that

1. Σ^F_α is a Γ-fullness preserving (α, λ^+)-iteration strategy for $\mathcal{P}$,
2. for $\alpha < \beta \leq \lambda^+$, $\Sigma^F_\alpha \subseteq \Sigma^F_\beta$,
3. for all $\alpha < \lambda^+$, whenever $\vec{\mathcal{T}}$ is according to Σ^F_α, $\vec{\mathcal{T}}^*$ is an F-quasi iteration of $\mathcal{P}$, and
4. if β is limit and $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ is a stack on $\mathcal{P}$ such that for each $\alpha < \beta$, $\oplus_{\xi < \alpha} \mathcal{T}_\xi$ is according to Σ^F_α then letting $\mathcal{M}$ be the direct limit along the main branch of $\vec{\mathcal{T}}$, $\mathcal{M}$ is suitable and $(\vec{\mathcal{T}}^*)^\frown \mathcal{M}$ is a F-quasi iteration of $\mathcal{P}$.

For $\alpha = 1$ we let $\Sigma^F_\alpha = \Lambda^F_{\mathcal{P}}$. Clearly, Σ^F_1 satisfies clauses 1-4 above. Suppose now we have defined $\langle \Sigma^F_\alpha : \alpha < \beta \rangle$. For $\alpha < \beta$, let I_α be the set of pairs $(\vec{\mathcal{T}}, \mathcal{M})$ such that $\vec{\mathcal{T}}$ is according to Σ^F_α, it has α many rounds, $\mathcal{M}$ is the last model of $\vec{\mathcal{T}}$, $\mathcal{M}$ is Γ-suitable and if α is limit then $\mathcal{M}$ is the direct limit of the models along the main branch. Suppose now $\beta = \nu + 1$ for some ν. Then $\Sigma^F_\beta(\vec{\mathcal{T}}) = b$ iff either

1. $\vec{\mathcal{T}}$ is a run of $\mathcal{G}(\mathcal{P}, \nu, \lambda^+)$, $\vec{\mathcal{T}}$ is according to Σ^F_ν and $\Sigma^F_\nu(\vec{\mathcal{T}}) = b$ or
2. $\vec{\mathcal{T}}$ is of the form $\vec{\mathcal{U}}^\frown \mathcal{M}^\frown \mathcal{U}$ such that $(\vec{\mathcal{U}}, \mathcal{M}) \in I_\nu$, $\mathcal{U}$ is according to $\Lambda^F_{\mathcal{M}}$ and $\Lambda^F_{\mathcal{M}}(\mathcal{U}) = b$.

Again, clearly Σ^F_β has the desired properties.

Lastly, suppose β is limit. We have that $\langle \Sigma^F_\alpha : \alpha < \beta \rangle$ satisfies 1-4. It is then enough to show that clause 4 holds. To see this, fix $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ as in clause 4. We have that for each $\alpha < \beta$, $(\vec{\mathcal{T}} \restriction \alpha)^*$ is an F-quasi iteration of $\mathcal{P}$. What we need to show is that the direct limit along the main branch of $\vec{\mathcal{T}}$ is the same as the direct limit defined by the two clauses of Definition 5.3. Let then, for $\alpha < \gamma < \beta$, $\pi_{\alpha,\gamma} : \mathcal{M}_\alpha \to \mathcal{M}_\gamma$ be the iteration embeddings given by $\vec{\mathcal{T}}$. We have that for each $f \in F$ and $\alpha < \gamma < \beta$, $\pi_{\alpha,\gamma}(f(\mathcal{M}_\alpha)) = f(\mathcal{M}_\gamma)$ and that $\mathcal{M}_\alpha = \cup_{f \in F} H_f^{\mathcal{M}_\alpha}$. Let then $\mathcal{M}$ be the direct limit of $\mathcal{M}_\alpha$ under the maps $\pi_{\alpha,\gamma}$ and let $\pi_\alpha : \mathcal{M}_\alpha \to \mathcal{M}$ be the direct limit embedding. We then have that $\mathcal{M} = \cup_{\alpha<\beta, f\in F}\pi_0(H_f^{\mathcal{P}})$ and for every $f \in F$, $\pi_0(H_f^{\mathcal{P}})$ is the direct limit under $\pi_{\alpha,\gamma}^{f,\beta}$ where $\langle \pi_{\alpha,\gamma}^{f,\xi} : \alpha < \gamma \leq \xi \leq \beta\rangle$ are the embeddings of $\vec{\mathcal{T}}^*$. It then follows that $\mathcal{M}$ is indeed the model constructed by the two clauses of Definition 5.3. □

Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. We then let $\Sigma_{\mathcal{P}}^F$ be the (λ^+, λ^+)-iteration strategy given by Lemma 6.5. Next we show that $\Sigma_{\mathcal{P}}^F$ has branch condensation.

Lemma 6.6 (ZF) *Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then $\Sigma_{\mathcal{P}}^F$ is a (λ^+, λ^+)-iteration strategy which is Γ-fullness preserving and has branch condensation.*

Proof. We only need to show that $\Sigma_{\mathcal{P}}^F$ has branch condensation as the rest follows from Lemma 6.5. Let then $(\mathcal{R}, \vec{\mathcal{T}}, \mathcal{Q}, \vec{\mathcal{U}}, c, \pi)$ be as in Definition 6.1 where we let $M = \mathcal{P}$ and $\Sigma = \Sigma_{\mathcal{P}}^F$. We need to see that $c = \Sigma_{\mathcal{P}}^F(\vec{\mathcal{U}})$. Because we have that $\pi : \mathcal{Q} \to \mathcal{R}$ and $\pi^{\vec{\mathcal{T}}} = \pi \circ \pi_c^{\vec{\mathcal{U}}}$ we get that, using clause 5 of Definition 6.3, that $\mathcal{Q}$ is Γ-suitable and $\pi_c^{\vec{\mathcal{U}}}(f(\mathcal{P})) = f(\mathcal{Q})$ for all $f \in F$. Let then $\langle \mathcal{U}_\alpha, \mathcal{M}_\alpha : \alpha \leq \eta\rangle$ be the normal components of $\vec{\mathcal{U}}$. Because $\vec{\mathcal{U}} \restriction \eta$ is via $\Sigma_{\mathcal{P}}^F$, we have that $\pi_{0,\eta}^{\vec{\mathcal{U}}}(f(\mathcal{P})) = f(\mathcal{M}_\eta)$ for every $f \in F$. It then follows that $\pi_c^{\mathcal{U}_\eta}(f(\mathcal{M}_\eta)) = f(\mathcal{Q})$ for all $f \in F$. Hence, $\mathcal{U}_\eta$ is according to $\Lambda_{\mathcal{M}_\eta}^F$, implying that $\Sigma_{\mathcal{P}}^F(\vec{\mathcal{U}}) = c$. □

We finish with the following lemma whose proof we leave to the reader as it is very close to the proof of Lemma 6.5.

Lemma 6.7 *Suppose F and G are two qsjs such that $F \subseteq G$. Then for any $\mathcal{P} \in S(\Gamma)$ witnessing that F is qsjs, $\mathcal{P}$ witnesses that G is also qsjs and $\Sigma_{\mathcal{P}}^F = \Sigma_{\mathcal{P}}^G$.*

7 (ω, Γ)-suitable premice

In this paper, our primary tool for constructing iteration strategies with branch condensation will be Lemma 6.5 which heavily relies on clause 4 and 5 of Definition 6.3. Here we develop some notions that we will later use to show that various F are qsjs and in particular, satisfy clause 4 and 5 of Definition 6.3.

We continue with the set up of the previous sections. Recall that we have fixed a cardinal λ and $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$. The basic notion we will need is that of (ω, Γ)-suitable premice. These are formed by stacking ω many Γ-suitable premice and hence, they all have ω-Woodin cardinals.

Definition 7.1 *A premouse $\mathcal{P}$ is (ω, Γ)-suitable if there is an increasing sequence of $\mathcal{P}$-cardinals $\langle \delta_i : i < \omega \rangle$ such that letting $\delta_\omega = \sup_{i<\omega} \delta_i$,*

1. *$\mathcal{P} \models$ "δ_ω is the largest cardinal",*
2. *for each i, δ_i is a Woodin cardinal in $\mathcal{P}$,*
3. *if $\mathcal{P}_i = \mathcal{P}|(\delta_i^{+\omega})^{\mathcal{P}}$ then $\mathcal{P}_0$ is a Γ-suitable premouse and $\mathcal{P}_{i+1}$ is a Γ-suitable premouse over $\mathcal{P}_i$.*

If $\mathcal{P}$ is (ω, Γ)-suitable then we let $\delta_i^{\mathcal{P}} = \delta_i$. We say $\mathcal{P}$ is an *anomalous* (ω, Γ)-suitable premouse if $\rho_\omega(\mathcal{P}) < \delta_\omega^{\mathcal{P}}$.

Suppose now $\mathcal{P}$ is any premouse with exactly ω-Woodin cardinals. Let $\langle \delta_i : i \in [-1, \omega) \rangle$ be such that $\delta_{-1} = \emptyset$ and $\langle \delta_i : i < \omega \rangle$ enumerates the Woodin cardinals of $\mathcal{P}$ in increasing order. Let $\delta_\omega = \sup_{i<\omega} \delta_i$. Suppose $\mathcal{P} \models$ "δ_ω is the largest cardinal". Let $\mathcal{T}$ be a normal tree on $\mathcal{T}$ constructed via $\mathcal{G}(\mathcal{P}, \lambda^+)$. Notice that there is a natural way of rearranging $\mathcal{T}$ so that it is constructed via a run of $\mathcal{G}(\mathcal{P}, \omega + 1, \lambda^+)$. In this rearrangement of $\mathcal{T}$, if $\mathcal{M}_n$ is the model of the beginning of the nth round and $n \leq \omega$ then the iteration embedding $i : \mathcal{P} \to \mathcal{M}_n$ exists. We have that $\mathcal{M}_0 = \mathcal{P}$. Furthermore, if $n < \omega$ and $\mathcal{T}_n$ is the tree played in the nth round then $\mathcal{T}_n$ is based on the window $(i(\delta_{n-1}), i(\delta_n))$. The tree played in the ωth round is played on $\mathcal{M}_\omega$ and is above $i(\delta_\omega)$. Notice that not every normal $\mathcal{T}$ will be constructed in exactly $\omega + 1$ non-trivial rounds (where we say that nth round is trivial if I doesn't play any extender from the window $(i(\delta_{n-1}), i(\delta_n))$).

In what follows, whenever we have a premouse like $\mathcal{P}$ above (and these include all (ω, Γ)-suitable premice), we will think of normal iteration trees on $\mathcal{P}$ as stacks produced via a run of $\mathcal{G}(\mathcal{P}, \omega + 1, \lambda^+)$. Given such a $\mathcal{P}$ and a normal tree $\mathcal{T}$ on $\mathcal{P}$, we say $\langle \mathcal{T}_n, \mathcal{M}_n : n < k \rangle$ are the normal components of $\mathcal{T}$ where $k \leq \omega + 1$ is the least such that for all $m \geq k$, $\mathcal{T}_m$ is undefined. We say $\mathcal{M}$ is *the last model* of $\mathcal{T}$ if

$$\mathcal{M} = \begin{cases} \mathcal{M}_\alpha^{\mathcal{T}_n} & : k = n + 1, \alpha + 1 = lh(\mathcal{T}_n) \\ \mathcal{M}_\omega & : k = \omega \text{ and } \mathcal{M}_\omega \text{ is the direct limit of } \mathcal{M}_n\text{'s} \\ \text{undefined} & : \textit{otherwise.} \end{cases}$$

Definition 7.2 *Suppose $\mathcal{P}$ is (ω, Γ)-suitable. We say Σ is an (ω, Γ)-fullness preserving strategy for $\mathcal{P}$ if Σ is an $(\omega^2, \lambda^+ + 1)$-iteration strategy such that whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ is a stack on $\mathcal{P}$ according to Σ the following holds:*

1. $\mathcal{M}_0 = \mathcal{P}$ *and for* $\alpha < \beta$, $\mathcal{M}_\alpha$ *is* (ω, Γ)*-suitable,*

2. *for* $\alpha < \beta$, *letting* $\langle \mathcal{T}_i^\alpha, \mathcal{M}_i^\alpha : i < k_\alpha \rangle$ *be the normal components of* $\mathcal{T}_\alpha$ *we have that*

   (a) $k_\alpha \leq \omega$,

   (b) *if* $\alpha + 1 < \beta$ *then* $\mathcal{M}_{\alpha+1}$ *is the last model of* $\mathcal{T}_\beta$,

   (c) *for each* $i < k_\alpha$ *and limit* $\xi < lh(\mathcal{T}_i^\alpha)$, $\mathcal{Q}(\mathcal{T}_i^\alpha \restriction \xi)$ *exists and*

$$\mathcal{Q}(\mathcal{T}_i^\alpha \restriction \xi) \trianglelefteq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T}_i^\alpha \restriction \xi)),$$

3. *if* $\beta = \omega^2$ *and* $\mathcal{M}_{\omega^2}$ *is the direct limit of* $\mathcal{M}_\alpha$ *under the iteration embeddings given by* $\vec{\mathcal{T}}$ *then* $\mathcal{M}_{\omega^2}$ *is* (ω, Γ)*-suitable.*

Next we need to introduce *simultaneous genericity iterations* which we will use to show that the strategies we construct have branch condensation. Simultaneous genericity iterations were used for this purpose in [6] as well. First, however, we need *generic genericity iterations*.

Suppose $N \in H_{\lambda^+}$ and $\mathcal{M} \in H_{\lambda^+}$ is a premouse with a Woodin cardinal δ. We say $\mathcal{T}$ is *the generic* N*-genericity tree* on $\mathcal{M}$ if $\mathcal{T}$ is a run of $\mathcal{G}(\mathcal{M}, \lambda^+)$ in which I plays as follows: at stage α, $E_\alpha^\mathcal{T}$ is the least extender such that for some $p \in Coll(\omega, N)$, p forces that if x is the generic code of N then x violates some axiom generated by $E_\alpha^\mathcal{T}$. If Σ is a $\lambda^+ + 1$-strategy for $\mathcal{M}$ then it can be shown that generic N-genericity iterations terminate and produce an iteration $i : \mathcal{M} \to \mathcal{Q}$ such that whenever $g \subseteq Coll(\omega, N)$ is generic and x is the generic code of N then x is generic over $\mathcal{Q}$ for the extender algebra at $i(\delta)$. The proof of this fact is just like the proof of the same fact for the usual genericity iterations (see [1] or [17]).

Definition 7.3 *We say* $\langle \mathcal{R}_i, \mathcal{Q}_i, m_i, \sigma_i, \nu_i, \Sigma : i < \omega \rangle$ *is a* Γ*-sequence of triangles with direct limit* $\mathcal{R}_\omega$ *if*

1. *for* $i \leq \omega$, $\mathcal{R}_i$ *and* $\mathcal{Q}_i$ *are* (ω, Γ)*-suitable premice,*

2. $m_i : \mathcal{R}_i \to_{\Sigma_1} \mathcal{R}_{i+1}$, $\sigma_i : \mathcal{Q}_i \to_{\Sigma_1} \mathcal{R}_{i+1}$ *and* $\nu_i : \mathcal{R}_i \to_{\Sigma_1} \mathcal{Q}_i$ *are such that* $m_i = \sigma_i \circ \nu_i$,

3. $\mathcal{R}_\omega$ *is the direct limit of* $\mathcal{R}_i$ *under the embeddings* $m_{i,j} =_{def} m_{j-1} \circ m_{j-2} \circ \cdots m_i$,

4. m_ω, σ_ω, *and* ν_ω *are undefined, and*

5. Σ *is an* (ω, Γ)*-fullness preserving strategy for* $\mathcal{R}_\omega$.

Suppose $\langle \mathcal{R}_i, \mathcal{Q}_i, m_i, \sigma_i, \nu_i, \Sigma : i \leq \omega\rangle$ is a Γ-sequence of triangles. Then we let $m_{i,\omega} : \mathcal{R}_i \to \mathcal{R}_\omega$ be the direct limit embedding and $\sigma_{i,\omega} = m_{i+1,\omega} \circ \sigma_i$. We then let Σ^i be $m_{i,\omega}$-pullback of Σ and Λ^i be $\sigma_{i,\omega}$-pullback of Σ.

Definition 7.4 (Simultaneous genericity iterations) *Suppose $\langle \mathcal{R}_k, \mathcal{Q}_k, m_k, \sigma_k, \nu_k, \Sigma : k < \omega\rangle$ is a Γ-sequence of triangles with direct limit $\mathcal{R}_\omega$ and $\vec{N} = \langle N_k : k < \omega\rangle \subseteq H_{\lambda^+}$. Suppose further that either $\lambda = \omega$ and Σ is an $(\omega^2, \omega_1 + 1)$-strategy or that $\lambda > \omega$ and for each i, $N_k \in H_\lambda$. We say $\langle \mathcal{R}^j_k, \mathcal{Q}^j_k, \vec{\mathcal{S}}^j_k, \vec{\mathcal{W}}^j_k, m^j_k, \sigma^j_k, \nu^j_k : j, k \leq \omega\rangle$ is the simultaneous $\vec{N}$-genericity iteration of $\langle \mathcal{R}_k, \mathcal{Q}_k : i < \omega\rangle$ via Σ if the following holds:*

1. *$\langle \mathcal{R}^0_k, \mathcal{Q}^0_k, m^0_k, \sigma^0_k, \nu^0_k : k < \omega\rangle = \langle \mathcal{R}_k, \mathcal{Q}_k, m_k, \sigma_k, \nu_k : k < \omega\rangle$,*
2. *for all $j, k < \omega$, $\vec{\mathcal{S}}^j_k$ is a non-dropping stack of finite length on $\mathcal{R}^j_k$ based on the window $(\delta^{\mathcal{R}^j_k}_j, \delta^{\mathcal{R}^j_k}_{j+1})$, $\vec{\mathcal{S}}^j_k$ is according to $\Sigma^k_{\mathcal{R}^j_k, \oplus_{l<j}\vec{\mathcal{S}}^l_k}$, and $\mathcal{R}^{j+1}_k$ is the last model of $\vec{\mathcal{S}}^j_k$,*
3. *for all $j, k < \omega$, $\vec{\mathcal{W}}^j_k$ is a non-dropping stack of finite length on $\mathcal{Q}^j_k$ based on the window $(\delta^{\mathcal{Q}^j_k}_j, \delta^{\mathcal{Q}^j_k}_{j+1})$, $\vec{\mathcal{W}}^j_k$ is according to $\Lambda^k_{\mathcal{Q}^j_k, \oplus_{l<j}\vec{\mathcal{W}}^l_k}$, and $\mathcal{Q}^{j+1}_k$ is the last model of $\vec{\mathcal{W}}^j_k$,*
4. *for $j, k < \omega$, $\sigma^j_k : \mathcal{Q}^j_k \to \mathcal{R}^j_{k+1}$, $m^j_k : \mathcal{R}^j_k \to \mathcal{R}^j_{k+1}$, $\nu^j_k : \mathcal{R}^j_k \to \mathcal{Q}^j_k$, and*
$$m^j_k = \sigma^j_k \circ \nu^j_k,$$
5. *for each j, $\vec{\mathcal{S}}^j_0$ is the tree of generic N_j-genericity iteration of $\mathcal{R}^j_0$ in which II plays according to $\Sigma^0_{\mathcal{R}^j_0, \oplus_{l<j}\vec{\mathcal{S}}^l_o}$,*
6. *for each $k, j < \omega$, letting $\vec{\mathcal{W}}^* = \nu^j_k \vec{\mathcal{S}}^j_k$, $\vec{\mathcal{W}}^j_k = \vec{\mathcal{W}}^{*\frown}\mathcal{W}$ and $\mathcal{M}$ be the last model of $\vec{\mathcal{W}}^*$, we have that $\mathcal{W}$ is the tree of generic N_j-genericity iteration of $\mathcal{M}$ which is based on the window $(\delta^{\mathcal{M}}_j, \delta^{\mathcal{M}}_{j+1})$ and is according to $\Lambda^k_{\mathcal{M}, (\oplus_{l<j}\vec{\mathcal{W}}^l_k)^\frown \vec{\mathcal{W}}^*}$,*
7. *for each $k, j < \omega$, letting $\vec{\mathcal{S}}^* = \sigma^j_k \vec{\mathcal{W}}^j_k$, $\vec{\mathcal{S}}^j_k = \vec{\mathcal{S}}^{*\frown}\mathcal{S}$ and $\mathcal{N}$ be the last model of $\vec{\mathcal{S}}^*$, we have that $\mathcal{S}$ is the tree of generic N_j-genericity iteration of $\mathcal{N}$ which is based on the window $(\delta^{\mathcal{N}}_j, \delta^{\mathcal{N}}_{j+1})$ and is according to $\Sigma^{k+1}_{\mathcal{N}, (\oplus_{l<j}\vec{\mathcal{S}}^l_k)^\frown \vec{\mathcal{S}}^*}$,*
8. *keeping the notation of clause 6 and 7, for each $k, j < \omega$, letting $s^j_k : \mathcal{R}^{j+1}_k \to \mathcal{M}$ and $w^j_k : \mathcal{Q}^{j+1}_k \to \mathcal{N}$ be the maps coming from the copying constructions, we have that $\nu^{j+1}_k = \pi^{\mathcal{W}} \circ s^j_k$, $\sigma^{j+1}_k = \pi^{\mathcal{S}} \circ w^j_k$ and $m^{j+1}_k = \sigma^{j+1}_k \circ \nu^{j+1}_k$.*

Suppose $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k < \omega \rangle$ is as in Definition 7.4. Then we say

$$\langle \mathcal{R}_k^\omega, \mathcal{Q}_k^\omega, \vec{\mathcal{S}}_k^\omega, \vec{\mathcal{W}}_k^\omega, m_k^\omega, \sigma_k^\omega, \nu_k^\omega : k < \omega \rangle$$

is the direct limit of $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k < \omega \rangle$ if

1. for each $k \leq \omega$, $\mathcal{R}_k^\omega$ is the direct limit of $\mathcal{R}_k^j$ under $\pi^{\vec{\mathcal{S}}_k^j}$'s,
2. for each $k < \omega$, $\mathcal{Q}_k^\omega$ is the direct limit of $\mathcal{Q}_k^j$ under $\pi^{\vec{\mathcal{W}}_k^j}$'s,
3. for each $k < \omega$, σ_k^ω, ν_k^ω and m_k^ω come from direct limit constructions.

Notice that we have that for each $k < \omega$, $\sigma_k^\omega : \mathcal{Q}_k^\omega \to \mathcal{R}_{k+1}^\omega$, $m_k^\omega : \mathcal{R}_k^\omega \to \mathcal{R}_{k+1}^\omega$, $\nu_k^\omega : \mathcal{R}_k^\omega \to \mathcal{Q}_k^\omega$, and

$$m_k^\omega = \sigma_k^\omega \circ \nu_k^\omega,$$

We also let $\mathcal{R}_\omega^\omega$ be the direct limit of $\mathcal{R}_k^\omega$ under m_k^ω. We say $\mathcal{R}_\omega^\omega$ is the direct limit of $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k \leq \omega \rangle$. Notice that, by the copying construction, $\mathcal{R}_\omega^\omega$ is a Σ-iterate of $\mathcal{R}_\omega$ and the length of the stack producing the iteration is ω^2.

8 Review of HOD analysis

In this section, we review HOD analysis of models satisfying $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + MC + \Theta = \theta_0$. Recall the definition of MC from Section 4. Until the end of this subsection, we assume V satisfies the above theory. We let $\Gamma = \mathcal{P}(\mathbb{R})$. For the duration of this subsection, we will drop Γ-from our notation. Thus, a suitable premouse is a Γ-suitable premouse and etc.

Suppose $\mathcal{P}$ is suitable and $A \subseteq \mathbb{R}$ is OD. We say $\mathcal{P}$ *weakly term captures* A if letting $\delta = \delta^{\mathcal{P}}$, for each $n < \omega$ there is a term relation $\tau \in \mathcal{P}^{Coll(\omega,(\delta^{+n})^{\mathcal{P}})}$ such that for comeager many $\mathcal{P}$-generics, $g \subseteq Coll(\omega, (\delta^{+n})^{\mathcal{P}})$, $\tau_g = \mathcal{P}[g] \cap A$. We say $\mathcal{P}$ term captures A if the equality holds for all generics. The following lemma is essentially due to Woodin and the proof can be found in [11].

Lemma 8.1 *Suppose $\mathcal{P}$ is suitable and $A \subseteq \mathbb{R}$ is OD. Then $\mathcal{P}$ weakly term captures A. Moreover, there is a suitable $\mathcal{Q}$ which term captures A.*

Given a suitable $\mathcal{P}$ and an OD set of reals A, we let $\tau_{A,n}^{\mathcal{P}}$ be the standard name for a set of reals in $\mathcal{P}^{Coll(\omega,(\delta^{+n})^{\mathcal{P}})}$ witnessing the fact that $\mathcal{P}$ weakly captures A. We then define $f_A \in F(\Gamma)$ by letting

$$f_A(\mathcal{P}) = \langle \tau^{\mathcal{P}}_{A,n} : n < \omega \rangle.$$

Let $F_{od} = \{f_A : A \subseteq \mathbb{R} \wedge A \in OD\}$.

All the notions we have defined in Section 3.1, Section 5 and Section 6 can be redefined for ordinal definable sets $A \subseteq \mathbb{R}$ using f_A as the relevant function. To save some ink, in what follows, we will say A-iterable instead of f_A-iterable and similarly for other notions. Also, we will use A in our subscripts instead of f_A.

The following lemma is one of the most fundamental lemmas used to compute HOD and it is originally due to Woodin. Again, the proof can be found in [11].

Theorem 8.2 *For each $f \in F_{od}$, there is $\mathcal{P} \in S(\Gamma)$ which is (F_{od}, f)-quasi iterable.*

Let $\mathcal{M}_\infty = \mathcal{M}_{\infty, F_{od}}$.

Theorem 8.3 (Woodin, [11]) $\delta^{\mathcal{M}_\infty} = \Theta$, $\mathcal{M}_\infty \in \mathrm{HOD}$ *and* $\mathcal{M}_\infty|\Theta = (V^{\mathrm{HOD}}_\Theta, \vec{E}^{\mathcal{M}_\infty|\Theta}, \in)$.

Finally, if $a \in H_{\omega_1}$, then we could define $\mathcal{M}_\infty(a)$ by working with suitable premice over a. Everything we have said about suitable premice can also be said about suitable premice over a and in particular, the equivalent of Theorem 8.3 can be proven using $\mathrm{HOD}_{a \cup \{a\}}$ instead of HOD and $\mathcal{M}_\infty(a)$ instead of $\mathcal{M}_\infty$.

9 The proof of Theorem 0.1

The rest of this paper is devoted to the proof of Theorem 0.1. From now on κ is as in the hypothesis of Theorem 0.1. Given $a \subseteq V_\kappa$, let

$$Lp(a) = \cup\{\mathcal{N} : \mathcal{N} \text{ is a sound countably iterable mouse over } a \text{ such that } \rho_\omega(\mathcal{N}) = a\}.$$

We define $\langle Lp_\xi(a) : \xi < \kappa^+ \rangle$ by the following recursion:

1. for $\alpha < \kappa^+$, $Lp_{\alpha+1} = Lp_1(Lp_\alpha(a))$,

2. for $\lambda < \kappa^+$ limit, $Lp_\lambda(a) = \cup_{\alpha<\lambda} Lp_\alpha(a)$.

Jensen's proof of square in L can be used to show that

Theorem 9.1 *for all $A \subseteq V_\kappa$, $Lp(A) \vDash \square_\kappa$.*

A proof of Theorem 9.1 can be found in [9]. We make the additional assumption that $|V_\kappa| = \kappa$. Otherwise, change V_κ to H_κ. From now on we fix $A \subseteq \kappa$ such that A codes V_κ. Because $\neg\Box_\kappa$, we must have that $o(Lp(A)) < \kappa^+$. Let $\xi = o(Lp(A))$. Because κ is singular, we have that $\mathrm{cf}(\xi) < \kappa$.

Definition 9.2 *We say $\mu < \kappa$ is a good point if $\mathrm{cf}(\kappa), \mathrm{cf}(\xi) < \mu$, μ is regular and $\mu^\omega = \mu$.*

Clearly there are good points. Suppose then that μ is a good point and $g \subseteq Coll(\omega, \mu)$ is generic. Working in $V[g]$, we let

$$\Gamma_{\mu,g} = \{B \subseteq \mathbb{R} : L(B, \mathbb{R}) \vDash AD^+\}.$$

The following theorem shows that $\Gamma_{\mu,g}$ is not empty. Its proof is essentially Steel's proof that $L(\mathbb{R}) \vDash AD$ (see [14]). See [6] for the definition of $L^\Sigma(\mathbb{R})$. It is the minimal Σ-mouse over $\mathbb{R}$ which contains all the ordinals and has no extenders on its sequence[2].

Theorem 9.3 *Suppose μ is a good point and $g \subseteq Coll(\omega, \mu)$ is generic. Suppose in $V[g]$, $\mathcal{M}$ is a κ^+-iterable countable mouse over some set X and that $\rho_\omega(\mathcal{M}) = X$. Let Σ be the κ^+-iteration strategy of $\mathcal{M}$. Then $L^\Sigma(\mathbb{R}) \vDash AD^+$. Hence, in $V[g]$, letting $\Lambda = \Sigma \restriction H_{\omega_1}$,*

$$Code(\Lambda) \in \Gamma_{\mu,g}.$$

Our goal is to show that for some good μ and generic $g \subseteq Coll(\omega, \mu)$, there is $B \in \Gamma_{\mu,g}$ such that

$$L(B, \mathbb{R}) \vDash AD^+ + \Theta = \theta_1.$$

We can then use Theorem 2.2 and the homogeneity of the forcing to prove Theorem 0.1. Recall the Minimality Assumption from Section 4. In what follows, we assume that it holds and derive a contradiction.

Borrowing some lemmas from the next subsection we can characterize sets in $\Gamma_{\mu,g}$ in terms of $\mathbb{R}$-mice.

Theorem 9.4 *Suppose μ is good and $g \subseteq Coll(\omega, \mu)$ is generic. Let $B \in \Gamma_{\mu,g}$. Then, in $V[g]$, there is a sound $\mathbb{R}$-mouse $\mathcal{N}$ such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$, $B \in \mathcal{N}$ and countable submodels of $\mathcal{N}$ are κ^+-iterable.*

[2]Below we stop feeding Σ after stage κ^+.

Proof. Fix g and B as in the hypothesis. We work in $V[g]$. We have that $L(B,\mathbb{R}) \vDash MC + \Theta = \theta_0$. It follows from the main theorem of [7] that $L(B,\mathbb{R}) \vDash V = L(Lp(\mathbb{R}))$. Hence, in $L(B,\mathbb{R})$, there is a countably iterable sound $\mathbb{R}$-mouse $\mathcal{N}$ such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$ and $B \in \mathcal{N}$. Fix such a mouse $\mathcal{N}$. It is then enough to show that, in $V[g]$, countable submodels of $\mathcal{N}$ are κ^+-iterable. To see this, let $\pi : \mathcal{S} \to \mathcal{N}$ be a countable submodel of $\mathcal{M}$. We have that $L(B,\mathbb{R}) \vDash$ "$\mathcal{S}$ is ω_1-iterable". It then follows from clause 2 of Lemma 9.10 (in particular, see clause 2c and 2d) that $\mathcal{S}$ is κ^+-iterable. □

Definition 9.5 *Suppose μ is good and $g \subseteq Coll(\omega,\mu)$ is generic. Working in $V[g]$, we let $\mathcal{S}^-_{\mu,g}$ be the union of those sound $\mathbb{R}$-mice $\mathcal{N}$ such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$, countable submodels of $\mathcal{N}$ are κ^+-iterable and there is a set of reals $B \in \Gamma_{\mu,g}$ such that B codes $\mathcal{N}$. We let $\mathcal{S}_{\mu,g} = L(\mathcal{S}^-_{\mu,g})$.*

Theorem 9.6 *Suppose μ is good and $g \subseteq Coll(\omega,\mu)$ is generic. Then in $V[g]$, $\mathcal{S}_{\mu,g} \vDash AD^+ + \theta_0 = \Theta$.*

Proof. It follows from the previous theorem that $\mathcal{P}(\mathbb{R})^{\mathcal{S}^-_{\mu,g}} = \Gamma_{\mu,g}$. Hence, $\mathcal{S}^-_{\mu,g} \vDash AD^+$. Also, notice that $\mathcal{S}^-_{\mu,g} \vDash \theta_0 = \Theta$. This is because otherwise there is $B \in \Gamma_{\mu,g}$ such that $w(B) = \theta_0^{\mathcal{S}^-_{\mu,g}}$. It then follows that $L(B,\mathbb{R}) \vDash \theta_0 < \Theta$ which contradicts (*). But now, using the scales analysis of [13] and [16] and the core model induction of [14], we get that $\mathcal{S}_{\mu,g} \vDash AD^+$. It then follows from (*) that $\mathcal{S}_{\mu,g} \vDash AD^+ + \theta_0 = \Theta$. □

9.1 Good points and good hulls

Clearly there are good points. Let μ be a good point. Recall that $\xi = o(Lp(A))$. Let $\nu = \mathrm{cf}(\xi)$ and let $f : \nu \to \xi$ be an increasing cofinal function. Let $\zeta = \kappa^{+\omega}$. We let $\Theta^\mu = \Theta^{S_{\mu,g}}$ and if $a \in H_{\mu^+}$ then we let $\mathcal{P}_{\mu,a} = (\mathcal{M}_\infty(a))^{S_{\mu,g}}$ where $g \subseteq Coll(\omega,\mu)$ is some generic. Notice that $\mathcal{P}_{\mu,a}$ is independent of g and $\mathcal{P}_{\mu,a} \in V$. We let $\mathcal{P}_\mu = \mathcal{P}_{\mu,\emptyset}$.

Definition 9.7 *We say (M,π) is a good hull at μ if $\pi : M \prec V_\zeta$ is such that $\mu + 1 \subseteq M$, $|M| = \mu$, $M^\omega \subseteq M$ and $\{A, f\} \in ran(\pi)$.*

An easy Skolem hull argument shows that there are good hulls at μ. Given a good hull (M,π) at μ and $a \in HC^{M[g]}$, we let $\mathcal{P}^M_{\mu,a} = \pi^{-1}(\mathcal{P}_{\mu,a})$ and $\mathcal{S}^M_{\mu,g} = (L(\pi^{-1}(\mathcal{S}_{\mu,g})))^{M[g]}$. If (M,π) is a good hull at μ then we let $\kappa_{M,\pi} = \pi^{-1}(\kappa)$ and $A_{M,\pi} = \pi^{-1}(A)$. Often times, when it is clear what π is, we will omit it from subscripts. The fact that M is countably closed implies that M is *full* with respect to countably iterable mice. The proof of this lemma is essentially the covering argument.

Lemma 9.8 *Suppose μ is good and (M, π) is a good hull at μ. Then $Lp(A_M) \in M$.*

Proof. We only outline the proof of this well-known fact. Suppose not. Let $\xi_M = \pi^{-1}(\xi)$. Let $\mathcal{M} \trianglelefteq Lp(A_M)$ be the least such that $\rho_\omega(\mathcal{M}) = A_M$ and $\mathcal{M} \notin M$. Let E be the (κ_M, κ)-extender from π and let $\mathcal{M}^* = Ult(\mathcal{M}, E)$. Then $\mathcal{M}^*$ is countably iterable (because E is countably closed) and is sound A-mouse such that $\rho_\omega(\mathcal{M}^*) = A$. Hence, $\mathcal{M}^* \trianglelefteq Lp(A)$. But because $\pi \restriction \xi_M$ is cofinal in ξ, we get that $Lp(A) \triangleleft \mathcal{M}^*$, contradiction. □

The following lemma is our main tool for extending iteration strategies to κ^+-iteration strategies. It is essentially due to Steel (see Lemma 1.25 of [14]) and we leave the proof to the readers.

Lemma 9.9 *Suppose μ is good and $g \subseteq Coll(\omega, \mu)$ is generic. Then the following holds.*

1. *Suppose $\mathcal{M}$ is a sound premouse over some set $X \in V_\kappa[g]$ such that $\rho_\omega(\mathcal{M}) = X$ and in $V[g]$, $\mathcal{M}$ is κ-iterable. Then $\mathcal{M}$ is κ^+-iterable.*

2. *Suppose $B \in \Gamma_{\mu,g}$ and $\Gamma \subset (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$ is a good pointclass. Suppose $(\mathcal{P}, \Lambda) \in V[g]$ is such that in $V[g]$, $\mathcal{P}$ is countable and Γ-suitable, and Λ is a Γ-fullness preserving (ω_1, ω_1)-iteration strategy for $\mathcal{P}$ with branch condensation. Then Λ can be extended to a (κ^+, κ^+)-iteration strategy which acts on non-dropping trees and has the branch condensation.*

Let Σ be the extension of Λ mentioned in clause 2 of Lemma 9.9. Then notice that it follows from branch condensation that if $\vec{\mathcal{T}}$ is a stack according to Σ and $\pi : N \to V_\zeta[g]$ is such that N is countable in $V[g]$ and $\vec{\mathcal{T}} \in rng(\pi)$ then $\pi^{-1}(\vec{\mathcal{T}})$ is according to Λ. Moreover, whenever $\mathcal{Q} \in I(\mathcal{P}, \Sigma)$, η is a cutpoint of $\mathcal{Q}$ and $\mathcal{M}$ is a sound mouse over $\mathcal{Q}|\eta$ projecting below η such that, in $V[g]$, its countable hulls have iteration strategy in $\Gamma_{\mu,g}$, then $\mathcal{M} \trianglelefteq \mathcal{Q}$. The following lemma will be used to show that good hulls are correct about κ^+-iterable mice.

Lemma 9.10 *Suppose μ is good and $g \subseteq Coll(\omega, \mu)$ is generic. Working in $V[g]$, suppose $B \in \Gamma_{\mu,g}$ and let $\Gamma^* = \mathcal{P}(\mathbb{R}) \cap L(B, \mathbb{R})$. Suppose that for some countable set X, $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma^*}(X)$ is such that $\rho_\omega(\mathcal{M}) = X$. Then, in $V[g]$, there are*

1. *a good pointclass*[3] $\Gamma \subset (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$,

2. *a* Γ*-suitable* $\mathcal{P}$ *over* X *which has an* (ω_1, ω_1)*-iteration strategy* $\Lambda \in L(B,\mathbb{R})$ *with branch condensation and a* (κ^+, κ^+)*-iteration strategy* Σ *with branch condensation which acts on non-dropping trees such that*

   (a) $L(B,\mathbb{R}) \vDash$ "Λ *is* Γ*-fulness preserving*",

   (b) Λ *and* Σ *agree on dom*$(\Sigma) \cap H_{\omega_1}$,

   (c) *there is some* α *such that if* $\mathcal{N}_\alpha$ *is the* α*th model of* $(L[\vec{E}][X])^{\mathcal{P}}$*-construction then* $Core(\mathcal{N}_\alpha) = \mathcal{M}$,

   (d) *for any* $\nu < \kappa$ *and for any* $h \subseteq Coll(\omega, \nu)$, Σ *can be extended to a* (κ^+, κ^+)*-iteration strategy in* $V[g * h]$ *and*

   (e) *if* $X \in V$ *then* $\mathcal{P} \in V$ *and* $\Sigma \restriction V \in V$.

Proof. Because $\mathcal{M}$ is iterable in $L(B,\mathbb{R})$, we can find a good pointclass $\Gamma \subset (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$ such that $\mathcal{M}$ has an iteration strategy coded into Γ. Let $\vec{C} \in L(B,\mathbb{R})$ be a Γ-sjs[4] and let $\Gamma_1 \subset (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$ be a good pointclass such that $\vec{C} \in \underset{\sim}{\Delta}_{\Gamma_1}$. Let $F : \mathbb{R} \to L(B,\mathbb{R})$ be a function such that for cone of x, $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures Γ_1 (as in Theorem 1.2.9 of [6]). Let $x \in \mathbb{R}$ be such that X is coded by x, $F(x)$ is defined and $\vec{C}$ is Suslin, co-Suslin captured by $(\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$. It follows that $\mathcal{W}^\Gamma(\mathcal{N}^*_x|\delta_x) \in \mathcal{N}^*_x$. Because sjs' condense to themselves, by a Skolem hull argument we get that there are club many η such that $\mathcal{W}^\Gamma(\mathcal{N}^*_x|\eta) \vDash$ "η is Woodin". Let then η be the least such that $\mathcal{W}^\Gamma(\mathcal{N}^*_x|\eta) \vDash$ "η is Woodin". It then follows that if $\mathcal{P}^- = (L[\vec{E}][X])^{\mathcal{N}^*_x|\eta}$ and $\mathcal{P} = Lp^\Gamma_\omega(\mathcal{P}^-)$ then $\mathcal{P}$ is Γ-suitable over X.

Appealing to universality (see Lemma 1.1.26 of [6]), we get that $\mathcal{M} \lhd \mathcal{P}$ and that it is reached by $(L[\vec{E}][X])^{\mathcal{P}}$-construction of $\mathcal{P}$. Working in $V[g]$, let $\Lambda \in L(B,\mathbb{R})$ be the (ω_1, ω_1)-iteration strategy of $\mathcal{P}$ induced by Σ_x (recall that background constructions inherit a strategy from the background as in [4]). It then follows that $L(B,\mathbb{R}) \vDash$ "Λ is Γ-fullness preserving" (one can use an argument like the one in Lemma 3.2.3 of [6]). Using Steel's lifting techniques from [14] which was summarized in clause 2 of Lemma 9.9, we can lift Λ to a (κ^+, κ^+)-iteration

[3] Recall a good pointclass is a pointclass which is closed under existential real quantification, is ω-parametrized and has the scale property.

[4] Recall that a Γ-sjs is a countable subset S of Γ such each $A \in S$ has a Γ-scale all of whose norms are in S, see [11] or [14] for more on sjs.

strategy Σ such that Σ acts on trees with no drops. We now have that $(\mathcal{P}, \Gamma, \Lambda, \Sigma)$ satisfies 1 and 2a-2c.

To get 2d, we can use generic comparisons. First, we show that Σ can be extended to a (κ, κ)-iteration strategy in $V[g * h]$ where for some $\nu < \kappa$, $h \subseteq Coll(\omega, \nu)$ is $V[g]$-generic. Working in $V[g]$, let $\pi : M \to V_\zeta[g]$ be a countable hull such that $(\mathcal{P}, \Sigma) \in rng(\pi)$. Let $\kappa_M = \pi^{-1}(\kappa)$ and let $\Sigma^M = \pi^{-1}(\Sigma)$. It is enough to show that 2d holds in M for κ_M and Σ^M. We have that $\Sigma^M = \Sigma \restriction V^M_{\kappa_M}$. Let $\nu < \kappa_M$ and let $k \subseteq Coll(\omega, \nu)$ be M-generic. Let

$$D_{\nu,k} = \{(a, \mathcal{W}^\Gamma(a)) : a \in V^{M[k]}_{\kappa_M}\}.$$

Let $C = \{(x, y) : x, y \in \mathbb{R} \wedge$ "y codes an $\mathcal{M} \lhd \mathcal{W}^\Gamma(x)$ such that $\rho_\omega(\mathcal{M}) = x$"$\}$. Without loss of generality, we can assume that Λ respects C (see Section 8 for the definition of "respecting" and also for the definition of $\tau^{\mathcal{P}}_C$ used below). Notice that if $i : \mathcal{P} \to \mathcal{Q}$ comes from an iteration according to Λ and $x \in \mathbb{R}$ is generic over $\mathcal{Q}$ for $\mathbb{B}^{\mathcal{Q}}$ then

$$\mathcal{W}^\Gamma(x) = \cup\{\mathcal{M} : \mathcal{M} \in \mathcal{Q}[x] \text{ is coded by } y \in \mathbb{R}^{\mathcal{Q}[x]} \text{ and } \mathcal{Q}[x] \vDash y \in \tau^{\mathcal{Q}}_C\} \qquad (1).$$

For each $\eta < \kappa_M$ let $\mathcal{P}_\eta \in M$ be the Σ^M-iterate of $\mathcal{P}$ which is obtained via the H^M_η-generic genericity iteration. Let $i_\eta : \mathcal{P} \to \mathcal{P}_\eta$ be the iteration embedding according to Σ^M.

It then follows from (1) that $D_{\nu,k} \in M[k]$. This is because $(a, \mathcal{N}) \in D_{\nu,k}$ iff whenever $\eta \in (\nu, \kappa_M)$, $l \subseteq Coll(\omega, \eta)$ is M-generic, x codes a and

$$\mathcal{S} = \cup\{\mathcal{M} : \mathcal{M} \in \mathcal{P}_\eta[x] \text{ is coded by } y \in \mathbb{R}^{\mathcal{P}_\eta[x]} \text{ and } \mathcal{P}_\eta[x] \vDash y \in i_\eta(\tau^{\mathcal{P}}_C)\}$$

then $\mathcal{N}$ is obtained from $\mathcal{S}$ via S-constructions (see [6] or [10]).

Because $D_{\nu,k} \in M[k]$ for all ν, k, we can do generic comparisons in M. Suppose $\nu < \kappa_M$ is an M-cardinal and $k \subseteq Coll(\omega, \nu)$ is M-generic. Then we let $\mathcal{S}_\eta \in M$ be the Σ^M-iterate of $\mathcal{P}$ which is obtained by generically comparing all Γ-suitable $\mathcal{Q}$'s which are in $H^{M[k]}_{\omega_1}$ (see [11] or [14]). We have that $\mathcal{S}_\eta \in M$ and whenever $l \subseteq Coll(\omega, \nu)$ is M-generic then $\mathcal{S}_\eta$ is the result of generically comparing all Γ-suitable premice that are in $H^{M[l]}_{\omega_1}$. Let $j_\eta : \mathcal{P} \to \mathcal{S}_\eta$ be the iteration map.

Fix now some $\nu < \kappa_M$ and let $h \subseteq Coll(\omega, \nu)$ be M-generic. Let $\vec{\mathcal{T}} \in H^{M[h]}_{\omega_1}$ be a stack on $\mathcal{P}$ which is according to Σ such that the last component of $\vec{\mathcal{T}}$ is of limit length. We let $\mathcal{Q}$ be the last model of $\vec{\mathcal{T}}$. Then $\mathcal{Q} \in M[h]$ is Γ-suitable. We then have that $\Sigma(\vec{\mathcal{T}}) = b$ iff b is the unique branch of $\vec{\mathcal{T}}$ such that $\mathcal{M}^{\mathcal{T}}_b = \mathcal{Q}$ and there is $\sigma : \mathcal{Q} \to \mathcal{S}_\nu$ such that

$$j_\nu = \sigma \circ \pi_b^{\vec{\mathcal{T}}}.$$

Let $\phi[\vec{\mathcal{T}}, \mathcal{Q}, b, \mathcal{S}_\nu]$ be the formula on the right side of the equivalence. It follows from absoluteness that $\Sigma(\vec{\mathcal{T}}) = b$ iff $M \vDash \phi[\vec{\mathcal{T}}, \mathcal{Q}, b, \mathcal{S}_\nu]$. Hence, $\Sigma \restriction V_{\kappa_M}^{M[g]} \in M[g]$. Using the elementarity of π we get that Σ can be extended to a (κ, κ)-strategy in $V[g * h]$ where for some $\nu < \kappa$, $h \subseteq Coll(\omega, \nu)$ is $V[g]$-generic. Σ can then be extended to a (κ^+, κ^+)-strategy by using clause 2 of Lemma 9.9. We then get that $(\mathcal{P}, \Gamma, \Lambda, \Sigma)$ also satisfies 2d. It remains to show that 2e can also be satisfied.

Notice that if $X \in V$ then $\mathcal{M} \in V$. Let then $\mathcal{R} = \mathcal{M}_\infty(\mathcal{P}, \Lambda)$. By homogeneity of the collapse, we have that $\mathcal{R} \in V$. Let now (M, π) be a good hull at μ such that $(\mathcal{R}, \Sigma_{\mathcal{R}}) \in rng(\pi)$[5] and $\mathcal{P}$ is countable in $M[g]$. Let $\mathcal{S} = \pi^{-1}(\mathcal{R})$. Then $\mathcal{S}$ is a Λ-iterate of $\mathcal{P}$ and $\mathcal{S} \in V$. It follows that $(\mathcal{S}, \Gamma, \Lambda_{\mathcal{S}}, \Sigma_{\mathcal{S}})$ satisfies 1 and 2a-2d. We claim that $(\mathcal{S}, \Sigma_{\mathcal{S}})$ satisfies 2e. The proof is just like the proof of 2d above using the fact that, by the homogeneity of the collapse, if G is the function $a \to \mathcal{W}^\Gamma(a)$ then $G \restriction V$ is definable in V. We leave the details to the reader. □

Suppose μ is good and $g \subseteq Coll(\omega, \mu)$ is generic. Working in $V[g]$, for $a \in V_\kappa[g]$ we let

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}^{(\mathcal{P}(\mathcal{P}(\kappa)))^{V[g]}}(a).$$

Lemma 9.11 *Suppose μ is a good point and $g \subseteq Coll(\omega, \mu)$ is generic. Then the following holds.*

1. *For any $a \in H_{\omega_1}^{V[g]}$, $\mathcal{W}_{\mu,g}(a) = \mathcal{W}^{\Gamma_{\mu,g}}(a)$.*
2. *Suppose (M, π) is a good hull at μ and $a \in V_{\kappa_M}^{M[g]}$. Then*

$$\mathcal{W}_{\mu,g}(a) \in M[g].$$

3. *Suppose $\nu > \mu$ is also good and let $h \subseteq Coll(\omega, \nu)$ be $V[g]$-generic. Suppose a is countable in $V[g]$. Then*

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}_{\nu,g*h}(a).$$

Proof.

[5] It is shown in [6] that if $\mathcal{P}$ is Γ-suitable, Σ is a Γ-fullness preserving iteration strategy for $\mathcal{P}$ with branch condensation and $\mathcal{R} \in I(\mathcal{P}, \Sigma)$ is an iterate of $\mathcal{P}$ via some $\vec{\mathcal{T}}$ then $\Sigma_{\mathcal{R},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. This is the reason for omitting the stack from the subscript in $\Sigma_{\mathcal{R}}$. We will do this throughout the paper.

1. It follows from Theorem 9.3 that $\mathcal{W}_{\mu,g}(a) \trianglelefteq \mathcal{W}^{\Gamma_{\mu,g}}(a)$. Suppose then $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. Let $B \in \Gamma_{\mu,g}$ be such that $L(B, \mathbb{R}) \vDash$ "$\mathcal{M}$ is ω_1-iterable". It then follows from 2c and 2d of Lemma 9.10 that $\mathcal{M}$ is κ^+-iterable. Hence, $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$.

2. We work in $V[g]$. First it follows from Lemma 9.8 that $\mathcal{W}_{\mu,g}(A_M) \in M$. It follows from S-constructions that $\mathcal{W}_{\mu,g}(A_M)[g] = \mathcal{W}_{\mu,g}(A_M, g) \in M[g]$. It then follows from clause 1 that for every $a \in V_{\kappa_M}^{M[g]}$, $\mathcal{W}_{\mu,g}(a) \in \mathcal{W}_{\mu,g}(A_M, g)$. Hence, clause 2 follows.

3. It follows from clause 1 that it is enough to show that $\mathcal{W}^{\Gamma_{\mu,g}}(a) = \mathcal{W}^{\Gamma_{\nu,h}}(a)$. Suppose then that $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. It follows from 2d of Lemma 9.10 that $\mathcal{M}$ is κ^+-iterable in $V[h]$. Hence, $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\nu,h}}(a)$. Conversely, suppose $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\nu,h}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. It then follows from 2e of Lemma 9.10 that $\mathcal{M}$ is κ^+-iterable in $V[g]$ implying that $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$.

□

Next we would like to show that good hulls capture κ^+-iterable mice, i.e., if (M, π) is good and $a \in H^M_{\kappa_M}$ then $\mathcal{W}_{\mu,g}(a) = (\mathcal{W}_{\mu,g}(a))^{M[g]}$. For this we will need the following lemma which will come handy later on as well.

Lemma 9.12 *Suppose μ is good and let $g \subseteq Coll(\omega, \mu)$ be generic. Then in $V[g]$, $\mathcal{P}_\mu$ is $\mathcal{P}(\mathcal{P}(\kappa))$-short tree iterable.*

Proof. Working in $\mathcal{S}_{\mu,g}$, let $\mathcal{Q}$ be a $\Gamma_{\mu,g}$-suitable premouse which is F_{od}-quasi-iterable. Let (M, π) be a good hull at μ such that $\mathcal{Q} \in M$ and $M[g] \vDash$ "$\mathcal{Q}$ is countable". By elementarity of π, it is enough to show that

$$M[g] \vDash \text{“}\mathcal{P}^M_\mu \text{ is } \mathcal{P}(\mathcal{P}(\kappa_M))\text{-short tree iterable”}.$$

We verify this only for trees. The proof for stacks is very similar and only notationally more involved. To show this for trees then it is enough to show that whenever $\mathcal{T} \in (H_{(\kappa^+_M)^M})^{M[g]}$ is a tree on $\mathcal{P}_\mu$ such that $M[g] \vDash$ "$\mathcal{T}$ is $\mathcal{P}(\mathcal{P}(\kappa_M))$-short" then there is $b \in M[g]$ such that b is a cofinal well-founded branch of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq (\mathcal{W}_{\mu,g}(\mathcal{M}(\mathcal{T})))^M$. Fix then such a tree $\mathcal{T}$ and let $\mathcal{N} = \mathcal{Q}(\mathcal{T})$. Notice that M is countable in $\mathcal{S}_{\mu,g}$ and in $\mathcal{S}_{\mu,g}$, $\mathcal{P}^M_\mu$ is an F_{od}-quasi-iterate of $\mathcal{Q}$. It then follows that

$$\mathcal{S}_{\mu,g} \vDash \text{“}\mathcal{P}^M_\mu \text{ is } \Gamma_{\mu,g}\text{-short tree iterable”} \qquad (1).$$

It follows from elementarity of π that $\pi(\mathcal{N})$ is κ^+-iterable in $V[g]$ and hence, $\mathcal{N}$ is κ^+-iterable in $V[g]$. Therefore, using clause 1 of Lemma 9.11, we get that

$$S_{\mu,g} \vDash \text{“}\mathcal{N} \text{ is } \omega_1\text{-iterable”} \qquad (2).$$

Combining (1) and (2), we get a cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{N} = \mathcal{Q}(b, \mathcal{T})$. By absoluteness $b \in M[g]$. □

Theorem 9.13 (Correctness of good hulls) *Suppose μ is good and $g \subseteq Coll(\omega, \mu)$ is generic. Suppose $a \in V_{\kappa_M}^{M[g]}$. Then*

$$\mathcal{W}_{\mu,g}(a) = (\mathcal{W}_{\mu,g}(a))^{M[g]}$$

Proof. Notice that because of π, $(\mathcal{W}_{\mu,g}(a))^{M[g]} \trianglelefteq \mathcal{W}_{\mu,g}(a)$. It is then enough to show that if $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$ then $M[g] \vDash$ “$\mathcal{M}$ is κ_M^+-iterable”. Because of Lemma 9.9, it is enough to show that $M[g] \vDash$ “$\mathcal{M}$ is κ_M-iterable”. It follows from clause 2 of Lemma 9.11 that $\mathcal{M} \in M$ (using homogeneity of the collapse).

Next, we have that the set

$$D = \{(x, \mathcal{W}_{\mu,g}(x)) : x \in V_{\kappa_M}^M\}$$

is OD_{A_M} and hence, by MC (recall the definition of MC from Section 4) in $\mathcal{S}_{\mu,g}$ and by clause 1 of Lemma 9.11, $D \in \mathcal{W}_{\mu,g}(A_M)$. Since $\mathcal{W}_{\mu,g}(A_M) \in M$, $D \in M$. Fix then some $\eta < \kappa_M$ such that $a \in H_\eta^M$. Let $\mathcal{Q}_\eta$ be the result of a generic genericity iteration of $\mathcal{P}_\mu^M$ making H_η generically generic. Notice that because $D \in M$ and $\mathcal{P}_\mu^M$ is $\Gamma_{\mu,g}$-short tree iterable we have that the entire iteration can be done in M. This iteration produces a $\Gamma_{\mu,g}$-correctly guided tree $\mathcal{T}$ with last model $\mathcal{Q}_\eta$ (where “last model” is in the sense of Definition 3.3).

Working in $\mathcal{S}_{\mu,g}$, let Λ be the ω_1-iteration strategy of $\mathcal{M}$. We have that whenever $\mathcal{Q}$ is $\Gamma_{\mu,g}$-suitable and $\mathcal{M}$ is generic over $\mathcal{Q}$ then $\Lambda \restriction H_{\omega_2}^{\mathcal{Q}[\mathcal{M}]} \in \mathcal{Q}[\mathcal{M}]$. It then follows that $\mathcal{Q}[\mathcal{M}] \vDash$ “$\mathcal{M}$ is ω_2-iterable”. We let $\Lambda^{\mathcal{Q}} = \Lambda \restriction H_{\omega_2}^{\mathcal{Q}[\mathcal{M}]}$. Notice that $\mathcal{Q}[\mathcal{M}]$ satisfies that “$\Lambda^{\mathcal{Q}}$ is the unique ω_2-iteration strategy of $\mathcal{M}$”.

We can now show that $\Lambda \restriction V_{\kappa_M}^{M[g]} \in M[g]$ as follows. Given an iteration tree $\mathcal{T} \in V_{\kappa_M}^{M[g]}$ on $\mathcal{M}$ which is according to Λ and has a limit length, we have that $\Lambda(\mathcal{T}) = b$ iff the following holds in $M[g]$: for all $\eta < \kappa_M$ such that $\mathcal{T} \in H_\eta^{M[g]}$, $\mathcal{T}$ is according to $\Lambda^{\mathcal{Q}_\eta}$ and $b = \Lambda^{\mathcal{Q}_\eta}(\mathcal{T})$. Hence, $\Lambda \restriction V_{\kappa_M}^{M[g]} \in M[g]$. □

The following are important corollaries.

Corollary 9.14 *Suppose μ is good and let $g \subseteq Coll(\omega, \mu)$ be generic. Let $\eta < o(\mathcal{P}_\mu)$. Then*

$$\mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta) = \mathcal{O}_\eta^{\mathcal{P}_\mu}.$$

Proof. Suppose first that $\mathcal{M} \trianglelefteq \mathcal{O}_\eta^{\mathcal{P}_\mu}$ is such that $\rho_\omega(\mathcal{M}) = \eta$. To show that $\mathcal{M} \trianglelefteq \mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta)$ it is enough to show that $\mathcal{M}$ is κ^+-iterable. To show this, it is enough to show the following claim.

Claim. Suppose μ is good and let $g \subseteq Coll(\omega, \mu)$ be generic. Let $\eta < o(\mathcal{P}_\mu)$ and let $\mathcal{M} \triangleleft \mathcal{O}_\eta^{\mathcal{P}_\mu}$ be such that $\rho_\omega(\mathcal{M}) = \eta$. Then, in $V[g]$, $\mathcal{M}$ is κ^+-iterable.
Proof. Let (M, π) be a good hull. It is enough to show the claim in M. Let then $\eta < o(\mathcal{P}_\mu^M)$ and let $\mathcal{M} \triangleleft \mathcal{O}_\eta^{\mathcal{P}_\mu^M}$ be such that $\rho_\omega(\mathcal{M}) = \eta$. We need to see that $\mathcal{M}$ is κ_M^+-iterable in M. Notice that since $\mathcal{P}_\mu^M$ is $\Gamma_{\mu,g}$-suitable, it follows from clause 1 of Lemma 9.11 that $\mathcal{M} \trianglelefteq \mathcal{W}^{\Gamma_{\mu,g}}(\mathcal{P}_\mu^M|\eta)$. It then follows from Lemma 9.13 that $\mathcal{M}$ is κ_M^+-iterable in M. □

Let now $\mathcal{M} \trianglelefteq \mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta)$ be such that $\rho(\mathcal{M}) = \eta$. We want to show that $\mathcal{M} \trianglelefteq \mathcal{P}_\mu$. Let (M, π) be a good hull such that $\mathcal{M} \in rng(\pi)$. Let $\mathcal{N} = \pi^{-1}(\mathcal{M})$. Notice that $\mathcal{N}$ is κ^+-iterable and hence, by Lemma 9.3, $\mathcal{S}_{\mu,g} \vDash$ "$\mathcal{N}$ is ω_1-iterable". Because $\mathcal{P}_\mu^M$ is $\Gamma_{\mu,g}$-suitable, we have that $\mathcal{N} \trianglelefteq \mathcal{P}_\mu^M$. Using elementarity of π we get that $\mathcal{M} \trianglelefteq \mathcal{P}_\mu$. □

Corollary 9.15 *Suppose $\mu < \nu$ are two good points. Suppose $g \subseteq Coll(\omega, \mu)$ is V-generic and $h \subseteq Coll(\omega, \nu)$ is $V[g]$-generic. Then for $a \in V_\kappa[g]$*

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}_{\nu,g*h}(a).$$

Proof. Using clause 2e of Lemma 9.10 applied in $V[g]$, we have that $\mathcal{W}_{\nu,g*h}(a) \trianglelefteq \mathcal{W}_{\mu,g}(a)$. Fix then $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$ such that $\rho_\omega(\mathcal{M}) = a$ and let (M, π) be a good hull at μ such that $\{\eta, \mathcal{M}\} \in rng(\pi)$. Let $\lambda = \pi^{-1}(\nu)$ and $\mathcal{N} = \pi^{-1}(\mathcal{M})$. It is enough to show that whenever $k \subseteq Coll(\omega, \lambda)$ is $M[g]$-generic, $\mathcal{N} \trianglelefteq (\mathcal{W}_{\lambda,g*k}(\pi^{-1}(a)))^{M[g][k]}$. Moreover, by genericity, it is enough to show this fact for $k \in V[g]$. Fix then such a k. The rest of the proof is then just a word-by-word generalization of the proof of Theorem 9.13 with $V = M[g][k]$. We leave it to the reader. □

Corollary 9.16 *Suppose $\mu < \nu$ are two good points. Suppose $g \subseteq Coll(\omega, \mu)$ is V-generic and $h \subseteq Coll(\omega, \nu)$ is $V[g]$-generic. Then for $a \in V_\kappa[g]$*

$$\mathcal{W}^{\Gamma_{\mu,g}}_{\mu,g}(a) = \mathcal{W}^{\Gamma_{\nu,g*h}}_{\nu,g*h}(a).$$

Proof. This follows immediately from clause 1 of Lemma 9.11 and Corollary 9.15. □

Let ν_0 be the least good point and let $h_0 \subseteq Coll(\omega, \nu_0)$. We let $W = V[h_0]$ and for $a \in H^{V[h_0]}_{\kappa^+}$ we let

$$\mathcal{W}(a) = \mathcal{W}^{(\mathcal{P}(\mathcal{P}(\kappa^+)))^W}(a).$$

In what follows, we will often deal with $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$-suitable premice. To make the notation convenient, we drop $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$ from our notation. Thus, in what follows, a suitable premouse is $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$-suitable, short tree is $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$-short and etc.

9.2 Excellent points and excellent hulls

Definition 9.17 *We say $\mu > \nu_0$ is excellent if μ is good and $\mu^{\nu_0} = \mu$.*

Definition 9.18 *Suppose μ is excellent and (M, π) is a good hull at μ. We say M is an excellent hull if $M^{\nu_0} \subseteq M$.*

Clearly if μ is excellent then there is an excellent hull at μ. Also, it is clear that there are at least ω many excellent points. The next two lemmas go back to Ketchersid's work done in [2]. Recall the definition of Θ^μ, which was defined in the first paragraph of Subsection 9.1. Also recall that $\Theta^\mu = \delta^{\mathcal{P}_\mu}$ (see Theorem 8.3).

Lemma 9.19 *Suppose μ is excellent. Then $\mathrm{cf}^V(\Theta^\mu) \leq \nu_0$.*

Proof. Suppose not and let $\pi : M \to V_\zeta$ be good at ν_0 such that $\mu \in ran(\pi)$. Let $\lambda = \pi^{-1}(\Theta^\mu)$. Because we are assuming $\mathrm{cf}^V(\Theta^\mu) > \nu_0$ and because $|\lambda| = \nu_0$, we have that

$$\nu =_{def} \sup(\pi \restriction \lambda) < \Theta^\mu.$$

Let $\mathcal{Q} = \pi^{-1}(\mathcal{P}_\mu)$. Let E be the $(crit(\pi), \nu)$-extender derived from $\pi \restriction \mathcal{Q} : \mathcal{Q} \to \mathcal{P}_\mu$. We let $\mathcal{N} = Ult(\mathcal{Q}, E_\pi \restriction \lambda)$ and let $j : \mathcal{Q} \to \mathcal{N}$ be the ultrapower embedding. Because $\delta^{\mathcal{Q}}$ is regular in $\mathcal{Q}$, we have that $j \restriction \delta^{\mathcal{Q}}$ is cofinal in $j(\delta^{\mathcal{Q}})$ and hence, $j(\delta^{\mathcal{Q}}) = \nu$. Also, we have that $\mathcal{P}_\mu | \nu = \mathcal{N} | \nu$.

Claim. $\mathcal{N} \trianglelefteq \mathcal{P}_\mu$.

Proof. Notice that if $(\mathcal{M}, k) \in V$ is such that $k : \mathcal{M} \to \mathcal{N}$ and $\mathcal{M}$ is countable in V then in W

there is $\sigma : \mathcal{M} \to \mathcal{Q}$. Hence, in $W = V[h_0]$, $\mathcal{M}$ is κ^+-iterable above $\sigma^{-1}(\delta^{\mathcal{Q}})$. It then follows that in W, $\mathcal{N}$ is countably iterable above ν. On the other hand, since $\nu < \Theta^\mu$, we have that $\mathcal{P}_\mu \vDash$ "ν isn't Woodin". Let then $\mathcal{N}^* \trianglelefteq \mathcal{P}_\mu$ be the least initial segment such that $\mathcal{J}_1(\mathcal{N}^*) \vDash$ "ν isn't Woodin". Then by Corollary 9.14, it follows that $\mathcal{N}^*$ is countably iterable. Therefore, because $\mathcal{N} \vDash$ "ν is Woodin", we have that $\mathcal{N} \trianglelefteq \mathcal{N}^* \trianglelefteq \mathcal{P}_\mu$. □

Let $\mathcal{S}$ be the largest initial segment of $\mathcal{P}_\mu$ such that $\mathcal{S} \vDash$ "ν is Woodin". Because $\mathcal{S} \in \mathrm{HOD}^{S_{\mu,h_0*h}}$ for any W-generic $h \subseteq Coll(\omega, \mu)$, it follows from the fact that MC holds in $\mathcal{S}_{\mu,h_0*h}$ that

$$\mathcal{P}(\delta^{\mathcal{Q}}) \cap L[\mathcal{S}, \mathcal{Q}] = (\mathcal{P}(\delta^{\mathcal{Q}}))^{\mathcal{Q}}.$$

It then also follows that we can lift j to

$$j^+ : L[\mathcal{S}, \mathcal{Q}] \to L[j^+(\mathcal{S}), \mathcal{N}].$$

Because E is countably closed, $j^+(\mathcal{S})$ is countably iterable in V. This means that $\mathcal{S} \in L[j^+(\mathcal{S}), \mathcal{N}]$ as it can be identified in $L[j^+(\mathcal{S}), \mathcal{N}]$ as the unique sound $\mathcal{N}$-premouse $\mathcal{M}$ such that $\rho_\omega(\mathcal{M}) = \nu$, $\mathcal{J}_1(\mathcal{M}) \vDash$ "ν is not Woodin" and in $L[j^+(\mathcal{S}), \mathcal{N}]^{Coll(\omega,\mathcal{N})}$, there is $\sigma : \mathcal{M} \to j^+(\mathcal{S})$. However, $L[j^+(\mathcal{S}), \mathcal{N}] \vDash$ "$j(\delta^{\mathcal{Q}}) = \nu$ is Woodin", contradiction! □

The proof of the lemma also gives the following, proof of which we leave to our readers.

Lemma 9.20 *Suppose μ is good. Then $\mathrm{cf}^V(\Theta^\mu) \leq \mu$.*

Let $T_\mu \in V$ be the tree on $\omega \times (\delta^2_1)^{\mathcal{S}_{\mu,g}}$ such that whenever $g \subseteq Coll(\omega, \mu)$ is generic,

$$S_{\mu,g} \vDash \text{"}p[T_\mu] \text{ is the universal } \Sigma^2_1\text{-set"}$$

where $p[T]$ is the projection of T. Next we show that realizable premice are suitable.

Lemma 9.21 *Suppose μ is an excellent point, (M, π) is an excellent hull at μ and $g \subseteq Coll(\omega, \mu)$ is W-generic. Let $H \in (V_{\kappa_M})^{M[h_0][g]}$ be such that $\mathcal{W}(H) \vDash$ "$H = H_\nu$ for some cardinal ν". Suppose further that $N \in V_{\kappa_M}^{M[h_0][g]}$ is such that there are embeddings $\sigma : \mathcal{W}(H) \to N$ and $\tau : N \to \pi(\mathcal{W}(H))$ such that $\pi \restriction \mathcal{W}(H) = \tau \circ \sigma$ and $\pi \restriction \mathcal{W}(H)$ is cofinal in $\mathcal{W}(H)$. Then*

$$N = \mathcal{W}(\sigma(H)).$$

Proof. Let $\eta > \mu$ be a good point such that for some λ, $\pi(\lambda) = \eta$ and $H, N \in V_\lambda^{M[h_0][g]}$. Let $T \in M$ be such that $\pi(T) = T_\eta$. Using Corollary 9.16, we have that $L[T, \mathcal{W}(H)] \models$ "$H = H_\nu$ for some cardinal ν". This implies that we can extend σ to act on $L[T, \mathcal{W}(H)]$. We let $\sigma^+ : L[T, \mathcal{W}(H)] \to L[\sigma^+(T), N]$ be this extension of σ. It then follows that we can find $\tau^+ : L[\sigma^+(T), N] \to L[\mathcal{T}_\eta, \pi(\mathcal{W}(H))]$ extending τ and such that

$$\pi \restriction L[T, \mathcal{W}(H)] = \tau^+ \circ \sigma^+.$$

Fix now an $n \in \omega$ such that whenever $h \subseteq Coll(\omega, \eta)$ is W-generic then in $\mathcal{S}_{\eta,h}$, $(T_\eta)_n$ projects to the set $\{(x, y) \in \mathbb{R}^2 : x$ codes a set a and an a-mouse $\mathcal{M}_x$ and y codes an a-mouse $\mathcal{M}_y$ such that $\rho_\omega(\mathcal{M}_y) = a$ and $\mathcal{M}_x \lhd \mathcal{M}_y\}$. Notice that the following holds in $L[T, \mathcal{W}(H)]$: for any generic $k \subseteq Coll(\omega, \mathcal{W}(H))$, in $L[T, \mathcal{W}(H)][k]$, for any $x \in \mathbb{R}$ coding $(H, \mathcal{W}(H))$ and for any real y, $(x, y) \notin p[T_n]$. We let this sentence be ϕ (in the language of $L[T, \mathcal{W}(H)]$).

We have that ϕ holds in $L[\sigma^+(T), N]$. Since $\tau : N \to \pi(\mathcal{W}(H))$, we also have that $N \trianglelefteq \mathcal{W}(\sigma(H))$. Moreover, since $\mathcal{P}(\mathcal{W}(H)) \cap L[T, \mathcal{W}(H)] \subseteq \mathcal{W}(\mathcal{W}(H))$, we get that, by the same argument as above, that $\mathcal{P}(N) \cap L[\sigma^+(T), N] \subseteq \mathcal{W}(N)$. It then follows that $\mathcal{P}(N) \cap L[\sigma^+(T), N]$ is countable in $W[g]$.

Suppose now that $N \neq \mathcal{W}(\sigma(H))$. Because we already know that $N \trianglelefteq \mathcal{W}(\sigma(H))$, it is enough to show that $\mathcal{W}(\sigma(H)) \trianglelefteq N$. Suppose not. There is then a κ^+-iterable sound $\mathcal{M} \trianglelefteq \mathcal{W}(\sigma(H))$ such that $\rho_\omega(\mathcal{M}) = \sigma(H)$ and $N \lhd \mathcal{M}$. Fix $k \subseteq Coll(\omega, \sigma(H))$ such that $k \in W[g]$ and k is both $M[h_0][g]$ and $L[\sigma^+(T), N]$ generic. Let $x \in (\mathbb{R})^{L[\sigma^+(T),N][k]}$ code $(\sigma(H), N)$ in a canonical fashion. We have that $x \in M[h_0][g][k]$. Let $y \in \mathbb{R}^{M[g][k]}$ code $\mathcal{M}$. We have that $(x, y) \in p[T_n]$. It then follows that $(x, y) \in p[\sigma^+(T_n)]$. Thus, by absoluteness, we can find $w \in L[\sigma^+(T), N][k]$ such that $(x, w) \in p[\sigma^+(T_n)]$. This, however, contradicts ϕ. □

The following is an immediate corollary of Lemma 9.21. We will use it to produce strategies which are fullness preserving.

Corollary 9.22 *Suppose μ is an excellent point, (M, π) is an excellent hull at μ and $g \subseteq Coll(\omega, \mu)$ is W-generic. Suppose $a \in H_{\omega_1}^{M[h_0]}$ and suppose further that $\mathcal{Q} \in H_{\kappa_M}^{M[h_0][g]}$ is an a-premouse such that there are Σ_1-elementary embeddings $\sigma : \mathcal{P}_{\mu,a}^M \to \mathcal{Q}$ and $\tau : \mathcal{Q} \to \mathcal{P}_{\mu,a}$ such that*

$$\pi \restriction \mathcal{P}_{\mu,a}^M = \tau \circ \sigma.$$

Then $\mathcal{Q}$ is suitable[6].

9.3 A quasi-sjs

Our ultimate goal is to isolate a quasi-self justifying system. Excellent hulls give possible candidates of such quasi-self justifying systems. Suppose μ is excellent and $g \subseteq Coll(\omega, \mu)$ is W-generic. Given an excellent μ, we let $F^{\mu} = F_{od}^{\mathcal{S}_{\mu,h_0*g}}$ (we drop the generics to save space). Let (M, π) be an excellent hull at μ. We then let $F^{M,\mu} = \pi^{-1}(F^{\mu})$ and $F^{M,\pi} = \pi[F^{M,\mu}]$. We will eventually show that $F^{M,\pi}$ is a quasi-sjs. The proof will be spread over several lemmas. Our first lemma is the following.

Lemma 9.23 *Suppose μ is excellent, $g \subseteq Coll(\omega, \mu)$ is W-generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then*

$$\mathcal{P}^M_{\mu,a} = \cup_{f \in F^{M,\pi}} H_f^{\mathcal{P}_{\mu,a}}.$$

Proof. This is an immediate consequence of the definition of $\mathcal{M}_\infty$. □

Lemma 9.24 *Suppose μ is excellent, $g \subseteq Coll(\omega, \mu)$ is W-generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in $\mathcal{S}_{\mu,h_0*g}$, $\mathcal{P}^M_{\mu,a}$ is $(F^{\mu}, F^{M,\pi})$-quasi iterable.*

Proof. Using Theorem 8.2 and elementarily of π, we get that for every $f \in F^{M,\pi}$ there is $\mathcal{Q} \in M[h_0][g]$ such that $\mathcal{S}_{\mu,h_0*g} \vDash$ "$\mathcal{Q}$ is (F^{μ}, f)-quasi iterable". It then follows that $\mathcal{P}^M_{\mu,a}$ is (F^{μ}, f)-quasi iterate of $\mathcal{Q}$. Since f was arbitrary, we get that in $\mathcal{S}_{\mu,h_0*g}$, $\mathcal{P}^M_{\mu,a}$ is (F^{μ}, f)-quasi iterable for every $f \in F^{M,\pi}$. It then follows that $\mathcal{S}_{\mu,h_0*g} \vDash$ "$\mathcal{P}_{\mu,a}$ is $(F^{\mu}, F^{M,\pi})$-quasi iterable".
□

If we restrict ourselves to stacks that are in $V^M_{\kappa_M}$ then we actually get that $F^{M,\pi}$ is a qsjs as witnessed by $\mathcal{P}_{\mu,a}$.

Lemma 9.25 ($F^{M,\pi}$ is a qsjs) *Suppose μ is excellent, $g \subseteq Coll(\omega, \mu)$ is W-generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in $\mathcal{S}_{\mu,h_0*g}$, $F^{M,\pi}$ is a qsjs as witnessed by $\mathcal{P}^M_{\mu,a}$ for stacks in $V^{M[h_0][g]}_{\kappa_M}$.*

[6] It follows from Lemma 9.19 that $\pi \restriction \mathcal{P}^M_{\mu,a}$ is cofinal in $\pi(\mathcal{P}^M_{\mu,a})$

Proof. We do the proof for $a = \emptyset$. Clearly $F^{M,\pi}$ and $\mathcal{P} = \mathcal{P}^M_\mu$ satisfy the first two clauses of Definition 6.3. We check clause 3 for quasi iterations that are in $V^{M[h_0][g]}_{\kappa_M}$. Let $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu \rangle \in V^{M[h_0][g]}_{\kappa_M}$ be an $(F^{M,\pi}, F^{M,\pi})$-quasi iteration of $\mathcal{P}$. Let $\langle \pi^{g,\alpha}_{\gamma,\xi} : \gamma < \xi < \alpha < \nu \rangle$ be the embeddings of $\vec{\mathcal{T}}$. We need to show the $\vec{\mathcal{T}}$ has a last model and for this we need to consider two cases. Suppose first $\nu = \alpha + 1$. Let $\mathcal{Q}$ be the last model of $\mathcal{T}_\alpha$. We need to show that $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H^{\mathcal{Q}}_f$.

Claim. $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H^{\mathcal{Q}}_f$.

Proof. Since $\mathcal{Q}$ is a $(F^{M,\pi}, F^{M,\pi})$-quasi iterate of $\mathcal{P}$, it is also $(F^\mu, F^{M,\pi})$-quasi iterate of $\mathcal{P}$. Hence, in $\mathcal{S}_{\mu,h_0 * g}$, $\mathcal{Q}$ is $F^{M,\pi}$-iterable. Let $\mathcal{S}$ be the transitive collapse of $\cup_{f \in F^{M,\pi}} H^{\mathcal{Q}}_f$ and let $\sigma : \mathcal{S} \to \mathcal{Q}$ be the inverse of the transitive collapse. Let $l = \pi_{\mathcal{Q},\infty,F^{M,\pi}}$. Then $l \circ \sigma : \mathcal{S} \to \pi(\mathcal{P})$. Because $\mathcal{P} = \cup_{f \in F^{M,\pi}} H^{\mathcal{P}}_f$, we have $k : \mathcal{P} \to \mathcal{S}$ such that

$$\pi \restriction P = l \circ \sigma \circ k.$$

Because $\mathcal{S} \in M$ (this follows from the fact that $\mathcal{S} \trianglelefteq \mathcal{Q}$) and $\pi \restriction \mathcal{P}$ is cofinal in $\pi(\mathcal{P})$ (see Lemma 9.19), it follows from Lemma 9.21 that $\mathcal{S}$ is suitable and hence, $\mathcal{S} = \mathcal{Q}$. It then follows that $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H^{\mathcal{Q}}_f$. We remark that the embedding l is used to apply Lemma 9.21. □

Next we assume ν is limit. In this case we have to verify that if for $f \in F^{M,\pi}$, H_f is the direct limit of $H^{\mathcal{M}_\gamma}_f$ under $\pi^{f,\nu}_{\gamma,\xi}$ then $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H_f$ is suitable. The proof is similar to the proof of the claim. We define $l : \mathcal{Q} \to \pi(\mathcal{P})$ as follows. Given $x \in \mathcal{Q}$ let α, f be such that $\alpha \in [\beta_{f,\nu}, \nu)$ and there is $\bar{x} \in H^{\mathcal{M}_\alpha}_f$ such that $\pi^{f,\nu}_{\alpha,\nu}(\bar{x}) = x$. Then let $l(x) = \pi_{\mathcal{M}_\alpha,\infty,f}(\bar{x})$. We then have $k : \mathcal{P} \to \mathcal{Q}$ coming from the direct limit construction and such that $\pi \restriction \mathcal{P} = l \circ k$. Therefore, using Lemma 9.21, we get that $\mathcal{Q}$ is suitable.

Actually, a little argument is needed to show that $\mathcal{Q} \in M$. To see this, first let $\vec{B} = \langle B_i : i < \omega \rangle \subseteq F^{M,\pi}$ be such that $\sup_{i<\omega} w(B_i) = \Theta^\mu$. Let $D_i = \{(x, y) : x \in \mathbb{R}^{M[h_0 * g]}, x$ codes a continuous function f and $y \in f^{-1}"B_i\}$. Then for any $\eta < \kappa_M$ and any $M[h_0 * g]$-generic $k \subseteq Coll(\omega, \eta)$ such that $k \in W[g]$, $\langle D_i \cap M[h_0 * g * k] : i < \omega \rangle \in M[h_0 * g * k]$. It is then easy to see that this is enough to compute $\mathcal{Q}$ in $M[h_0 * g]$.

As part of the proof of clause 3 we have also shown clause 4 of Definition 6.3. We leave the details to the reader. To show clause 5, let $\mathcal{Q} \in V^{M[h_0][g]}_{\kappa_M}$ be a $(F^{M,\pi}, F^{M,\pi})$-quasi iterate of $\mathcal{P}$ and $\sigma : \mathcal{R} \to_{\Sigma_1} \mathcal{Q}$ be such that $f(\mathcal{Q}) \in rng(\sigma)$ for all $f \in F^{M,\pi}$ and $\mathcal{R} \in M$. We need to see that $\mathcal{R}$ is suitable. Again, we have $rng(\pi_{\mathcal{P},\mathcal{Q},F^{M,\pi}}) \subseteq rng(\sigma)$ and hence, there is an embedding

$k : \mathcal{P} \to_{\Sigma_1} \mathcal{R}$ such that $\pi_{\mathcal{P},\mathcal{Q},F^{M,\pi}} = \sigma \circ k$. Let then $l = \pi_{\mathcal{Q},\infty,F^{M,\pi}} \circ \sigma$. It follows that $\pi \restriction \mathcal{P} = l \circ k$ and hence, $\mathcal{R}$ is suitable. □

Next, we show that we can substitute $F^{M,\pi}$ by a subset of it which is countable in W. We make this move as we believe it makes the arguments that follow more transparent. We could just as well carry on with $F^{M,\pi}$.

Definition 9.26 (Pre-sjs) *Suppose μ is excellent and $g \subseteq Coll(\omega, \mu)$ is W-generic. We say $\vec{B} = \langle B_i : i < \omega \rangle$ is a pre-sjs at μ if for every $a \in H_{\omega_1}^{W[g]}$.*

1. *for every $i < \omega$, $B_i \in F^\mu$,*

2. *for every $n < \omega$ there is $k < \omega$ such that $B_k = \oplus_{i<n} B_i$,*

3. *$\mathcal{P}_{\mu,a} = \cup_{i<\omega} H_{B_i}^{\mathcal{P}_{\mu,a}}$.*

Lemma 9.27 *Suppose μ is excellent and $g \subseteq Coll(\omega, \mu)$ is W-generic. Then there is a pre-sjs at μ.*

Proof. It is enough to prove that for some excellent (M, π), $M \vDash$ "there is a pre-sjs at μ". Fix then some excellent (M, π). We have that $M[h_0] \vDash \text{cf}(\delta^{\mathcal{P}^M_\mu}) = \omega$. Fix then some increasing sequence $\langle \gamma_i : i < \omega \rangle \in M$ cofinal in $\delta^{\mathcal{P}^M_\mu}$. Let $\delta = \delta^{\mathcal{P}^M_\mu}$. We can then find $\vec{B}^* = \langle B_i^* : i < \omega \rangle \subseteq F^{M,\pi}$ such that for every $a \in H_{\omega_1}^{M[h_0 * g]}$, $\sup_{i<\omega} \gamma_{B_i}^{\mathcal{P}^M_{\mu,a}} = \delta^{\mathcal{P}^M_{\mu,a}}$. Next, close $\vec{B}^*$ under $\oplus$ operator to obtain $\vec{B} = \langle B_i : i < \omega \rangle$. We claim that $M \vDash$ "$\pi^{-1}(\vec{B})$ is a pre-sjs".

To see this, it is enough to show that for every a, $\mathcal{P}^M_{\mu,a} = \cup_{i<\omega} H_{B_i}^{\mathcal{P}^M_{\mu,a}}$. Fix $n < \omega$ and let $\mathcal{T}$ on $\mathcal{P}^M_{\mu,a}|(\delta^+)^{\mathcal{P}^M_{\mu,a}}$ be the correctly guided maximal tree for making $\mathcal{P}^M_{\mu,a}|(\delta^{+n-1})^{\mathcal{P}^M_{\mu,a}}$ generic (here, we let $\delta^{+-1} = \delta$). We have that $\mathcal{T} \in \mathcal{P}^M_{\mu,a}$ and $\delta(\mathcal{T}) = (\delta^{+n})^{\mathcal{P}^M_{\mu,a}}$. Also, because $\mathcal{P}^M_{\mu,a}$ is $(F^{M,\pi}, F^{M,\pi})$-quasi iterable, we have a branch b of $\mathcal{T}$ such that $\mathcal{M}_b^\mathcal{T}$ is suitable and $\pi_b^\mathcal{T}(\delta) = \delta(\mathcal{T})$. Moreover, letting $\mathcal{Q} = \mathcal{M}_b^\mathcal{T}$, for every $k, m < \omega$, $\pi_b^\mathcal{T}(\tau_{B_k,m}^{\mathcal{P}^M_{\mu,a}}) = \tau_{B_k,m}^\mathcal{Q}$.

Notice now that for each $i < \omega$, $\pi_{\mathcal{P}^M_{\mu,a},\mathcal{Q},B_i} \in H_{B_i}^{\mathcal{P}^M_{\mu,a}}$. This follows from the fact that $\pi_{\mathcal{P}^M_{\mu,a},\mathcal{Q},B_i}$ only depends on $\mathcal{P}^M_{\mu,a}$, $\mathcal{Q}$ and the term relations capturing B_i over $\mathcal{P}^M_{\mu,a}$ and $\mathcal{Q}$. This information is coded into $H_{B_i}^{\mathcal{P}^M_{\mu,a}}$. $\mathcal{S}$ is the transitive collapse of $\cup_{i<\omega} H_{B_i}^{\mathcal{P}^M_{\mu,a}}$ and $\sigma : \mathcal{S} \to \mathcal{P}^M_{\mu,a}$ is the inverse of the collapse then $\sigma \restriction (\delta^{+n})^\mathcal{S}$ is cofinal implying that $\sigma = id$. □

Suppose μ is excellent, (M, π) is an excellent hull at μ, $g \subseteq Coll(\omega, \mu)$ is V-generic and $\vec{B} \in V[g]$ is a pre-sjs at μ. We let $B_i^M =_{def} B_i \cap M[g]$ and $\vec{B}^M =_{def} \langle B_i^M : i < \omega \rangle$. We say

(M,π) *captures* $\vec{B}$ if $\vec{B}^M \in M$ and $\pi(\vec{B}^M) = \vec{B}$. We let $F_{\vec{B}} = \{f_{B_i} : i < \omega\}$. It follows from Lemma 9.25 and the proof of Lemma 9.27 that

Lemma 9.28 *Suppose μ is excellent, $g \subseteq Coll(\omega,\mu)$ is W-generic, (M,π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in $\mathcal{S}_{\mu,h_0*g}$, $F_{\vec{B}}$ is a qsjs as witnessed by $\mathcal{P}^M_{\mu,a}$ for stacks in $V^{M[h_0][g]}_{\kappa_M}$.*

We continue with the set up of the lemma. Given any $B \in \mathcal{P}(\mathbb{R}^{W[g]}) \cap OD^{\mathcal{S}_{\mu,h_0*g}}$, we let

$$f^{M,g}_B = \{(\mathcal{Q}, f_B(\mathcal{Q})) : \mathcal{Q} \in V^{M[g]}_{\kappa_M} \text{ and } \mathcal{Q} \text{ is } \Gamma_{\mu,g}\text{-suitable}\}.$$

We define f^M_B similarly. Given a pre-sjs $\vec{B}$ at μ such that M captures $\vec{B}$, we let

$$f^{M,g}_{\vec{B}} = \langle f^{M,g}_{B_i} : i < \omega\rangle \text{ and } f^M_{\vec{B}} = \langle f^M_{B_i} : i < \omega\rangle.$$

Lemma 9.29 *Suppose μ is excellent, (M,π) is an excellent hull at μ, $g \subseteq Coll(\omega,\mu)$ is W-generic and $\vec{B} \in V[g]$ is a pre-sjs at μ captured by (M,π). Then $f^{M,g}_{\vec{B}} \in M[h_0 * g]$.*

Proof. We have that in $\mathcal{S}_{\mu,h_0*g}$, for each i, $f^{M,g}_{B_i}$ is $OD_{V^{M[h_0][g]}_{\kappa_M}}$ and hence, $f^{M,g}_{B_i} \in M[h_0][g]$ (recall that $\mathcal{W}(A_M) \in M$). Let then for each $i < \omega$, $\dot{f}_i \in M[h_0]^{Coll(\omega,\mu)}$ be the term that is always realized as $f^{M,g}_{B_i}$. Because $M[h_0]$ is ω-closed in W, we have that $\langle \dot{f}_i : i < \omega\rangle \in M[h_0]$. Hence, $f^{M,g}_{\vec{B}} \in M$. □

Suppose now μ is excellent, $g \subseteq Coll(\omega,\mu)$ is W-generic, $\vec{B}$ is a pre-sjs at μ, and (M,π) is an excellent hull at μ capturing $\vec{B}$. Suppose further that $a \in (H_{\omega_1})^{M[g]}$. Lemma 9.25 and Lemma 6.5 imply that we can define $\Sigma^{F_{\vec{B}}}$ for $\mathcal{P}^M_{\mu,a}$. In our case, however, we can only restrict to iterations that are in $V^{M[h_0][g]}_{\kappa_M}$. We then let $\Sigma^{M,g}_{\mu,\vec{B},a}$ be this iteration strategy.

Lemma 9.30 *$\Sigma^{M,g}_{\mu,\vec{B},a}$ is a (κ_M,κ_M)-iteration strategy which acts on iterations that are in $M[h_0][g]$ and is fullness preserving. Moreover, if $\vec{\mathcal{T}} \in V^{M[h_0][g]}_{\kappa_M}$ is via $\Sigma^{M,g}_{\mu,\vec{B},a}$ and the last normal component of $\vec{\mathcal{T}}$ has limit length then $\Sigma^{M,g}_{\mu,\vec{B},a}(\vec{\mathcal{T}}) \in M[h_0][g]$.*

Proof. The first part is an immediate consequence of Lemma 6.5. The second part follows from the fact that $f^{M,g}_{\vec{B}} \in M[h_0][g]$. □

Notice that $\Sigma^{M,g}_{\mu,\vec{B},a} \restriction V^{M[h_0]}_{\kappa_M} \in M[h_0]$ given that $a \in M[h_0]$. We let $\Sigma^M_{\mu,\vec{B},a} = \Sigma^{M,g}_{\mu,\vec{B},a} \restriction V^{M[h_0]}_{\kappa_M}$. When $a = \emptyset$, we omit it from subscripts. In the next subsection we will show that for each $i < \omega$ there is some $\Sigma^{M,g}_{\mu,\vec{B},a}$-iterate $\mathcal{Q}$ of $\mathcal{P}^M_{\mu,a}$ such that if Λ is the strategy of $\mathcal{Q}$ induced by $\Sigma^{M,g}_{\mu,\vec{B},a}$ then Λ strongly respects B_i. It then easily follows from Lemma 6.6 that Λ has branch condensation.

9.4 An ω-suitable $\mathcal{P}$

Recall that we have that $\mathrm{cf}^W(\kappa) = \omega$. Let then $\langle \mu_i : i < \omega \rangle$ be a sequence of excellent points such that for each i, $\mu_{i+1}^{\mu_i^+} = \mu_{i+1}$ and $\sup_{i<\omega} \mu_i = \kappa$. For the rest of the paper, we fix a W-generic $g_0 \subseteq Coll(\omega, \mu_0)$ and a pre-sjs $\vec{B}$ at μ_0.

Fix now an excellent (M^*, π^*) at μ_0 capturing $\vec{B}$. For each $X \in V_{\kappa_{M^*}}^{M^*[h_0][g_0]}$ we let $\mathcal{T}_X$ be the tree on $\mathcal{P}_{\mu_0}^{M^*}$ according to $\Sigma_{\mu_0,\vec{B}}^{M^*}$ that makes X generic. Let $\mathcal{Q}_X$ be the last model of $\mathcal{T}_X$ and let $\pi_X : \mathcal{P}_{\mu_0}^{M^*} \to \mathcal{Q}_X$ be the iteration embedding according to $\Sigma_{\mu_0,\vec{B}}^{M^*}$. We then let $\tau_{B_i,X} = \pi_X(\tau_{B_i,0}^{\mathcal{P}_{\mu_0}^{M^*}})$. We let

$$\tau_i = \{(X, \mathcal{Q}_X, \tau_{B_i,X}) : X \in V_{\kappa_{M^*}}^{M^*[h_0][g]}\}$$

and let $\vec{\tau} = \langle \tau_i : i < \omega \rangle$. The proof of Lemma 9.29 gives that $\vec{\tau} \in M^*[h_0][g]$.

To continue, we introduce the following notation. Suppose $\mu \leq \nu$ are two excellent points and (M, π) and (N, σ) are two excellent hulls at μ and ν respectively. We write $(M, \pi) < (N, \sigma)$ if $(M, \pi) \in rng(\sigma)$. In this case, we let $\pi_{M,N} = \sigma^{-1}(\pi) : M \to N$. We have that $\pi_{M,N} \in N$.

Our first goal here is to show that we can extend $\vec{B}$ to a pre-sjs at any other excellent point $> \mu_0$. Given an excellent $\mu > \mu_0$, we let $B_{\mu,i} \in W[g_0]^{Coll(\omega,\mu)}$ be the name for the set of reals given by $\pi^*(\tau_i)$ as follows: given a standard name for a real $\dot{x} \in W[g_0]^{Coll(\omega,\mu)}$ and $p \in Coll(\omega, \mu)$,

$$(p, \dot{x}) \in B_{\mu,i} \leftrightarrow W[g_0] \vDash \text{“ if } \eta = \mu^{++} \text{ and } (\mathcal{Q}, \tau) \text{ is such that } (V_\eta, \mathcal{Q}, \tau) \in \pi^*(\tau_i) \text{ then } p \Vdash_{Coll(\omega,\mu)}^{\mathcal{Q}[V_\eta]} \dot{x} \in \tau\text{”}.$$

If $g \subseteq Coll(\omega, \mu)$ is $W[g_0]$ generic then we let $B_{\mu,i,g}$ be the realization of $B_{\mu,i}$. We let $\vec{B}_\mu = \langle B_{\mu,i} : i < \omega \rangle$ and $\vec{B}_{\mu,g} = \langle B_{\mu,i,g} : i < \omega \rangle$.

Lemma 9.31 *Suppose $\mu > \mu_0$ and $g \subseteq Coll(\omega, \mu)$ is $W[g_0]$-generic. Then $B_{\mu,i,g} \in \mathcal{S}_{\mu,h_0*g_0*g}$.*

Proof. It is enough to show that $L(B_{\mu,i,g}, \mathbb{R}^{W[g_0*g]}) \vDash AD^+$. Let (M, π) be a good hull at μ_0 such that $(M^*, \pi^*) < (M, \pi)$ and $\mu \in rng(\pi)$. Let $B = \pi^{-1}(B_{\mu,i})$. It is enough to show that if $\lambda = \pi^{-1}(\mu)$ and $h \subseteq Coll(\omega, \lambda)$ is $M[h_0 * g_0]$-generic such that $h \in W[g_0]$ then in $M[h_0 * g_0 * h]$, $L(B_h, \mathbb{R}) \vDash AD^+$ where B_h is the realization of B in $M[h_0 * g_0 * h]$. Fix then such an h. Notice that we have that

$$B_h = B_i \cap \mathbb{R}^{M[h_0*g_0*h]}.$$

Since B_i is OD in $\mathcal{S}_{\mu_0,h_0*g_0}$, we have that there is a sound $\mathbb{R}^{M[h_0*g_0*h]}$-mouse $\mathcal{M}$ such that $B_h \in \mathcal{M}$, $\rho_\omega(\mathcal{M}) = \mathbb{R}^{M[h_0*g_0*h]}$ and $\mathcal{S}_{\mu_0,h_0*g_0} \vDash$ "$\mathcal{M}$ is ω_1-iterable". It then follows from Lemma 9.11 and Lemma 9.13 that $\mathcal{M} \in M[h_0 * g_0 * h]$ and $M[h_0 * g_0 * h] \vDash$ "$\mathcal{M}$ is κ_M^+-iterable". Therefore, $\mathcal{M} \trianglelefteq (\mathcal{S}^-_{\lambda,h_0*g_0*h})^{M[h_0*g_0*h]}$ implying that $M[h_0 * g_0 * h] \vDash L(B_h, \mathbb{R}) \vDash AD^+$. □

Lemma 9.32 *Suppose $\mu > \mu_0$ and $g \subseteq Coll(\omega, \mu)$ is $W[g_0]$-generic. Then $\vec{B}_{\mu,i,g}$ is a pre-sjs at μ.*

Proof. The proof is like the proof of Lemma 9.31. It follows from the proof of Lemma 9.27 that it is enough to show that for any $a \in H_{\omega_1}^{W[g_0*g]}$, $\sup_{i<\omega} \gamma^{\mathcal{P}_{\mu,a}}_{B_{\mu,i,g}} = \Theta^\mu$. We first show this for $a = \emptyset$. Let (M, π) be an excellent hull at μ_0 such that $(M^*, \pi^*) < (M, \pi)$ and $\mu \in rng(\pi)$. Let $\lambda = \pi^{-1}(\mu)$ and let $h \subseteq Coll(\omega, \lambda)$ be $M[h_0 * g_0]$-generic such that $h \in W[g_0]$. It is enough to show that in $M[h_0 * g_0 * h]$,

$$\sup_{i<\omega} \gamma^{\mathcal{P}}_{B_{\mu,i,g}} = \delta^{\mathcal{P}}$$

where $\mathcal{P} = \pi^{-1}(\mathcal{P}_\mu)$. This follows from the fact that $(B_{\mu,i,g})^{M[h_0*g_0*h]} = B_i \cap M[h_0 * g_0 * h]$ and in M, $\mathcal{P}$ is a $F_{\vec{B}}$-quasi-iterate of $\mathcal{P}^M_{\mu_0}$.

To show the claim for every a use Lemma 9.25 and Lemma 9.27. □

We define $\langle \mathcal{P}_i : i < \omega \rangle$ as follows:

1. $\mathcal{P}_0 = \mathcal{P}_{\mu_0}$,

2. $\mathcal{P}_{i+1} = \mathcal{P}_{\mu_{i+1},\mathcal{P}_i}$.

Let $\delta_i = \delta^{\mathcal{P}_i}$.

Lemma 9.33 $\mathcal{P}_{i+1} \vDash$ "*δ_i is Woodin" and no level of $\mathcal{P}_{i+1}$ projects across δ_i.*

Proof. It is enough to show that no level of $\mathcal{P}_{i+1}$ projects across δ_i. Towards a contradiction, suppose $\mathcal{M} \trianglelefteq \mathcal{P}_{i+1}$ is such that $\rho(\mathcal{M}) < \delta_i$. It follows from Corollary 9.14 that $\mathcal{M}$, regarded as a mouse over $\mathcal{P}_i$, is κ^+-iterable in W and hence, $\mathcal{M} \trianglelefteq \mathcal{P}_i$, contradiction. □

Let $\delta_\omega = \sup_{i<\omega} \delta_i$ and $\mathcal{P}^- = \cup_{i<\omega} \mathcal{P}_i$ and let

$$\mathcal{P} = \begin{cases} \mathcal{W}(\mathcal{P}^-) & : \text{ if no level of } \mathcal{W}(\mathcal{P}^-) \text{ projects across } \delta_\omega \\ \mathcal{N} & : \text{ where } \mathcal{N} \triangleleft \mathcal{W}(\mathcal{P}^-) \text{ is the least such that } \rho(\mathcal{N}) < \delta_\omega \end{cases}$$

Notice that $\kappa = \delta_\omega$. In subsequent sections we will show that $\mathcal{P} = \mathcal{W}(\mathcal{P}^-)$. Before we move on, we fix some notation.

Suppose (M, π) is an excellent hull at μ_k for some $k \leq \omega$. Let $\mu_{-1} = \nu_0$. We say (M, π) is *perfect* if $(M^*, \pi^*) < (M, \pi)$, $M^{\mu_{k-1}^+} \subseteq M$ and $\langle \mu_i : i < \omega \rangle \in rng(\pi)$. Notice that we have that $\mathcal{P} \in rng(\pi)$. We then let $\langle \mu_i^M : i \in \omega \rangle = \pi^{-1}(\langle \mu_i : i \in \omega \rangle)$, $\mathcal{P}_i^M = \pi^{-1}(\mathcal{P}_i)$ and $\mathcal{P}^M = \pi^{-1}(\mathcal{P})$.

Notice that if for some $k < \omega$, $g \subseteq Coll(\omega, \mu_k)$ is $W[g_0]$-generic and (M, π) is a perfect hull at μ_k then $\Sigma^{M, g_0 * g}_{\mu_k, a, \vec{B}_{\mu_k, g}}$ is defined for every $a \in (H_{\omega_1})^{M[h_0 * g_0 * g]}$. We then let $\Sigma^{M,g}_{\mu_k, a} = \Sigma^{M, g_0 * g}_{\mu_k, a, \vec{B}_{\mu_k, g}}$ where we let $\mathcal{P}_{-1} = \emptyset$. Notice that we have that $\Sigma^{M,g}_{\mu_k, a} \restriction M[h_0 * g_0] \in M[h_0 * g_0]$. We then let $\Sigma^M_{\mu_k, a} = \Sigma^{M,g}_{\mu_k, a} \restriction M[h_0 * g_0]$. It is again true that $\Sigma^M_{\mu_k, a} \in M[h_0 * g_0]$.

9.5 Iteration strategy for $\mathcal{P}$

Suppose $(R, \tau) > (M^*, \pi^*)$ is a perfect hull at some μ_p for $p \leq \omega$. Here we will describe a (κ_R, κ_R)-strategy for $\mathcal{P}^R$ which acts on stacks all of whose normal components are in $V^R_{\kappa_R}$ and are above $\delta^{\mathcal{P}^R}_{p-1}$ (recall that $\delta_{-1} = 0$). To do this, we introduce the auxiliary game $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ in which player II, after a notification that I is about to start a new round, will play an embedding $\pi_\alpha : \mathcal{M}_\alpha \to \mathcal{P}$ such that $\tau \restriction \mathcal{P} = \pi_\alpha \circ \pi_{0,\alpha}$ where $\pi_{0,\alpha}$ is the iteration embedding. More precisely, $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha, \pi_\alpha : \alpha < \eta \rangle \subseteq V^R_{\kappa_R}$ is a run of $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ in which neither player has lost if

1. for $\alpha < \eta$, $\mathcal{M}_\alpha$ is the model at the beginning of the αth round and $\mathcal{M}_0 = \mathcal{P}^R$,

2. for $\alpha < \eta$, $\mathcal{M}_\alpha$ is (anomalous) ω-suitable premouse,

3. for $\alpha < \eta$, $\mathcal{T}_\alpha$ is a tree with no fatal drops based on some window $(\delta_k^{\mathcal{M}_\alpha}, \delta_{k+1}^{\mathcal{M}_\alpha})$ for $k \geq p-1$ (see Definition 7.1 and the line below it for the explonation of "anomalous") ,

4. for $\alpha < \eta$, $\mathcal{T}_\alpha$ is correctly guided and if $\mathcal{T}_\alpha$ has a last branch and $\mathcal{T}_\alpha^-$ is short then $\mathcal{Q}(\mathcal{T}_\alpha^-) \trianglelefteq \mathcal{M}^{\mathcal{T}_\alpha}_{lh(\mathcal{T}_\alpha)-1}$,

5. $\pi_0 = \pi \restriction \mathcal{P}^R$ and for $\alpha < \beta < \eta$, $\pi_\alpha : \mathcal{M}_\alpha \to \mathcal{P}$,

6. if for $\alpha < \beta < \eta$, $\pi_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the iteration embedding then for $\alpha < \beta < \eta$, $\pi_\alpha = \pi_\beta \circ \pi_{\alpha,\beta}$.

The game has at most ω^3 many rounds. As usual player I plays the extenders and starts new rounds. II plays branches. I cannot start a new round if the current model isn't suitable. If I

decides to start a new round with $\mathcal{M}_\alpha$ as its starting model then II has to play the embedding π_α that satisfies clauses 5 and 6. The game stops if one of the models produced is ill founded or if II cannot play π_α. II wins if all the models appearing in the iteration are wellfounded, she can always play the embeddings π_α, the game runs ω^3 many rounds or if one of the rounds runs κ_R steps.

Lemma 9.34 *Player II has a wining strategy in $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$.*

Proof. We define the strategy Λ for II by induction. First fix some bijection $f : \kappa_R \to V^R_{\kappa_R}$ such that $f \in R$. In the first round, I plays a tree on $\mathcal{P}^R = \mathcal{M}_0$. Suppose I starts playing the first round on $(\delta_k^{\mathcal{M}_0}, \delta_{k+1}^{\mathcal{M}_0})$ for $k \geq p-1$ and suppose $\mathcal{T}$ is a tree constructed according to Λ and $\mathcal{T}$ has limit length. Below we describe how II should play her move.

1. If $\mathcal{T}$ is short then II plays the unique cofinal branch $b \in R$ of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$ exists,

2. if $\mathcal{T}$ is maximal then II plays b such that

   (a) $b \in R$,

   (b) $\mathcal{M}_b^{\mathcal{T}}$ is ω-suitable,

   (c) there is $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{P}$ such that $\tau \restriction \mathcal{P}^R = \sigma \circ \pi_b^{\mathcal{T}}$.

   (d) b is the f-least branch satisfying 2a-2c.

We need to see that II can satisfy these conditions along with satisfying the rules of the iteration game. It is enough to show that there is some $b \in R$ satisfying all the clauses above except 2d. We can then choose f-least such b.

Claim. There is $b \in R$ satisfying 1 and 2a-2c.
Proof. Fix some perfect hull (M, π) at μ_{k+1} such that $(R, \tau) < (M, \pi)$.

Subclaim. $\mathcal{T}$ is according to $\pi_{R,M}$-pullback of $\Sigma^M_{\mu_{k+1}, \mathcal{P}_k^M}$.
Proof. Fix limit $\alpha < lh(\mathcal{T})$. By induction, we assume that $\mathcal{T} \restriction \alpha$ is according to $\pi_{R,M}$-pullback of $\Sigma^M_{\mu_{k+1}, \mathcal{P}_k^M}$. Notice that $\pi_{R,M}\mathcal{T} \restriction \alpha \in M$. To prove the subclaim, we need to see that if b is the branch of $\mathcal{T} \restriction \alpha$ then b is also the branch of $\pi_{R,M}\mathcal{T} \restriction \alpha$. Notice that we must have that $\mathcal{T} \restriction \alpha$

isn't maximal and hence, $\mathcal{Q}(\mathcal{T}\restriction\alpha)$ exists and $\mathcal{Q}(\mathcal{T}\restriction\alpha)\trianglelefteq\mathcal{M}^{\mathcal{T}}_{\alpha}$. Let $c=\Sigma^M_{\mu_{k+1},\mathcal{P}^M_k}(\pi_{R,M}\mathcal{T}\restriction\alpha)$. It follows that

$$\mathcal{Q}(c,\pi_{R,M}\mathcal{T}\restriction\alpha)\text{ exists iff }\mathcal{Q}(c,\mathcal{T}\restriction\alpha)\text{ exists}$$

implying that if $b\neq c$ then $\mathcal{Q}(c,\mathcal{T}\restriction\alpha)$ doesn't exist. We then have that $\pi_{R,M}\mathcal{T}\restriction\alpha$ is a maximal tree. Let $\mathcal{U}=\pi_{R,M}\mathcal{T}\restriction\alpha$. Because $\mathcal{U}\in M$, it follows from the construction of $\Sigma^M_{\mu_{k+1},\mathcal{P}^M_k}$ that there is a map $\sigma:\mathcal{M}^{\mathcal{U}}_c\to\mathcal{P}$ such that $\pi\restriction\mathcal{P}^M=\sigma\circ\pi^{\mathcal{U}}_c$. Let $k:\mathcal{M}^{\mathcal{T}\restriction\alpha}_c\to\mathcal{M}^{\mathcal{U}}_c$ be the copy map. It then follows that $\tau\restriction\mathcal{P}^R=\sigma\circ k\circ\pi^{\mathcal{T}\restriction\alpha}_c$ which implies, using Lemma 9.22, that $\mathcal{M}^{\mathcal{T}\restriction\alpha}_c$ is ω-suitable. Hence, $\mathcal{Q}(\mathcal{T}\restriction\alpha)$ cannot exist, contradiction! □

We then let $b=\Sigma^M_{\mu_{k+1},\mathcal{P}^M_k}(\pi_{R,M}\mathcal{T})$. It follows from the proof of the subclaim that b satisfies clause 1. Suppose then $\mathcal{T}$ is maximal. It follows from the copying construction that $\pi_{R,M}\mathcal{T}$ is also maximal. It also follows from the construction of $\Sigma^M_{\mu_{k+1},\mathcal{P}^M_k}$ that there is $\sigma:\mathcal{M}^{\pi_{R,M}\mathcal{T}}_b\to\mathcal{P}$ such that $\pi\restriction\mathcal{P}^M=\sigma\circ\pi^{\pi_{R,M}\mathcal{T}}_b$. Letting $l:\mathcal{M}^{\mathcal{T}}_b\to\mathcal{M}^{\pi_{R,M}\mathcal{T}}_b$ come from the copying construction, we have that $\tau\restriction\mathcal{P}^R=\sigma\circ l\circ\pi^{\mathcal{T}}_b$. Hence, $\mathcal{M}^{\mathcal{T}}_b$ is ω-suitable. Moreover, b satisfies clause 2b-2c. To finish, we need to show that $b\in R[h_0*g_0]$. Recall that $R[h_0*g_0]$ is ω-closed in $W[g_0]$ and that in $W[g_0]$, $\mathrm{cf}(\delta^{\mathcal{P}^R}_{k+1})=\omega$. Let then $\langle\gamma_i:i<\omega\rangle\subseteq rng(\pi^{\mathcal{T}}_b)$ be cofinal in $\delta(\mathcal{T})$. Then we have that $\langle\gamma_i:i<\omega\rangle\in R$. We also have that b is the unique branch c of $\mathcal{T}$ such that $\langle\gamma_i:i<\omega\rangle\subseteq\pi^{\mathcal{T}}_c$. It then follows from absoluteness that $b\in R$. □

We then define $\Lambda(\mathcal{T})=b$ which satisfies clause 1 and 2 above. Suppose now that we have defined Λ for stacks that have $<\alpha$-many rounds. We want to extend Λ to act on stacks that have α-many rounds. The first step is to specify II's move at the beginning of round α. To do this, we examine three cases. Suppose first α is limit. Let $\langle\mathcal{T}_\xi,\mathcal{M}_\xi,\pi_\xi:\xi<\gamma<\alpha\rangle$ be a run of $\mathcal{G}^a(\mathcal{P}^R,\omega^3,\kappa_R)$ which is according to Λ. Then I must start the αth round on $\mathcal{M}_\alpha$ which is the direct limit of $\mathcal{M}_\xi$ under the iteration embeddings $\pi_{\xi,\gamma}:\mathcal{M}_\xi\to\mathcal{M}_\gamma$. Let $\pi_{\xi,\alpha}:\mathcal{M}_\xi\to\mathcal{M}_\alpha$ be the direct limit embedding. We then let $\pi_\alpha:\mathcal{M}_\alpha\to\mathcal{P}$ come from the direct limit construction: given $x\in\mathcal{M}_\alpha$, let ξ be such that for some $\bar{x}\in\mathcal{M}_\xi$, $x=\pi_{\xi,\alpha}(\bar{x})$ and let

$$\pi_\alpha(x)=_{def}\pi_\xi(\bar{x}).$$

We then let II play π_α. It follows that π_α is as desired and that $\mathcal{M}_\alpha$ is ω-suitable.

Next suppose $\alpha=\beta+1$ and let $\langle\mathcal{T}_\xi,\mathcal{M}_\xi,\pi_\xi:\xi<\gamma\leq\beta\rangle$ be a run of $\mathcal{G}^a(\mathcal{P}^R,\omega^3,\kappa_R)$ which is according to Λ and suppose I wants to start the αth round of the game. Then I has to start the round on the last model $\mathcal{M}_\alpha$ of $\mathcal{T}_\beta$. It follows from our inductive hypothesis that $\mathcal{M}_\alpha$ is ω-suitable. It remains to choose π_α. Suppose first that $\mathcal{T}_\beta^-$ is defined and is maximal. Let b be the last branch of $\mathcal{T}_\beta$. As part of our inductive hypothesis we have that there is $\pi : \mathcal{M}_b^{\mathcal{T}_\beta^-} \to \mathcal{P}$ such that $\pi_\beta = \pi \circ \pi_b^{\mathcal{T}^-}$. In this case, we let $\pi_\alpha = \pi$.

Next suppose that either $\mathcal{T}_\beta^-$ isn't defined or that it is defined but it is short. Recall the meaning of μ_k, which were defined in the first paragraph of Subsection 9.4. Let $k < \omega$ be such that $\mathcal{T}_\beta$ is a tree played on the window $(\delta_k^{\mathcal{M}_\beta}, \delta_{k+1}^{\mathcal{M}_\beta})$. Also, let $(M, \pi) > (R, \tau)$ be a perfect hull at μ_{k+1}. It follows from the proof of the subclaim that $\mathcal{T}_\beta$ is according to $\pi_{R,M}$-pullback of $\Sigma^M_{\mu_{k+1}, \mathcal{P}_k^M}$. Let then $l : \mathcal{M}_\alpha \to \mathcal{N}$ come from the copying construction where $\mathcal{N}$ is the last model of $\pi_{R,M}\mathcal{T}_\beta$. By the construction of $\Sigma^M_{\mu_{k+1}, \mathcal{P}_k^M}$ we have $\sigma : \mathcal{N} \to \mathcal{P}$ such that $\pi \restriction \mathcal{P}^M = \sigma \circ \pi^{\pi_{R,M}\mathcal{T}_\beta}$. It then follows that $\pi_\beta = \sigma \circ l \circ \pi^{\mathcal{T}_\beta}$. Let then $\pi_\alpha = \sigma \circ l$.

Next, to describe II's moves in the αth round we just follow the steps describing II's move in the first round. More precisely, given a tree $\mathcal{T} \in R$ on $\mathcal{M}_\alpha$ which is according to Λ and has limit length, we let II play as follows:

1. If $\mathcal{T}$ is short then II plays the unique cofinal branch $b \in R$ of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$ exists,

2. if $\mathcal{T}$ is maximal then II plays b such that

   (a) $b \in R$,

   (b) $\mathcal{M}_b^{\mathcal{T}}$ is ω-suitable,

   (c) there is $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{P}$ such that $\pi_\alpha = \sigma \circ \pi_b^{\mathcal{T}}$.

   (d) b is the f-least branch satisfying 2a-2c.

That there is such a branch $b \in R$ follows from the proof of the claim above. This finishes our description of Λ. □

We let Λ^R be the winning strategy of $\mathcal{P}^R$ in the iteration game $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ constructed above. Next, we show that perfect hulls of $\mathcal{P}$ are iterable. More precisely, suppose $(M^*, \pi^*) < (R, \tau)$ is a perfect hull at μ_p for some $p < \omega$. Let $\mathcal{S} \in R$ be an ω-suitable Λ^R iterate of $\mathcal{P}^R$ and let $k : \mathcal{S} \to \mathcal{P}$ be the realization according to Λ. Let $(M, \pi) > (R, \tau)$ be a perfect hull at μ_{p+1} such that $k \in rng(\pi)$. Notice that $H_{\mu_p^{++}} \in M$. We let $\Lambda^{R,M,\mathcal{S}}$ be the k^*-pullback of Λ^M where

$k^* = \pi^{-1}(k)$. Because $H_{\mu_p^{++}} \in M$, we have that $\Lambda^{R,M,\mathcal{S}}$ is a $(0, \omega^3, \mu_p^{++})$-strategy for $\mathcal{S}$ acting on stacks above $\delta_p^{\mathcal{S}}$.

Lemma 9.35 $\mathcal{P} = \mathcal{W}(\mathcal{P}_\omega^-)$.

Proof. Suppose not and let k be least such that $\rho_\omega(\mathcal{P}) < \delta_k$. Suppose (R, τ) is a perfect hull at μ_{k+1}. Let $\mathcal{Q} = \mathcal{P}^R$ and let p be the standard parameter of $\mathcal{Q}$. Let $\mathcal{N} = Hull_m^{\mathcal{Q}}(\delta_{k+1}, \{p\})$ where m is the least such that $\rho_m(\mathcal{Q}) < \delta_\omega^{\mathcal{Q}}$. We claim that $\Sigma^R_{\mu_{k+1}, \mathcal{P}_k^R}$ has branch condensation. To see this, it is enough to show that if $\vec{\mathcal{T}}$ is a stack on $\mathcal{N}$ according to $\Sigma^R_{\mu_{k+1}, \mathcal{P}_k^R}$ such that $\pi^{\vec{\mathcal{T}}}$-exists and $\mathcal{S}$ is the last model of $\vec{\mathcal{T}}$ then $\mathcal{S}$ is (m, μ_{k+1}^{++})-iterable above $\delta_k^{\mathcal{S}}$. But choosing M as in the paragraph proceeding the lemma, we see that $\Lambda^{R,M,\mathcal{S}}$ is such a strategy for $\mathcal{S}$.

It then follows that $\Sigma =_{def} \tau(\Sigma^R_{\mu_{k+1}, \mathcal{P}_k^R})$ is a fullness preserving (κ, κ)-strategy for $\mathcal{P}_{k+1}$ acting on trees above δ_k. Hence, Σ can be extended to a (κ^+, κ^+)-iteration strategy (see Lemma 9.9). Let then $h \subseteq Coll(\omega, \mu_{k+2})$ be W-generic. It then follows from Theorem 9.3 that in $W[g_0 * h]$, $L^\Sigma(\mathbb{R}) \vDash AD^+$ implying that letting $\Sigma^* =_{def} \Sigma \restriction H_{\omega_1}^{W[g_0*h]}$, $\Sigma^* \in S_{\mu_{k+2}, h_0 * g_0 * h}$.

Notice next that Σ^* is S_{μ_{k+2}, h_0*g_0*h}-fullness preserving strategy (see Theorem 9.3). Because $S_{\mu_{k+2}, h_0*g_0*h} \vDash MC$, it follows that, in $W[g_0 * h]$, $L(\Sigma^*, \mathbb{R}) \vDash AD^+ + \theta_0 < \Theta$, contradiction! See Remark 4.2 for an explanation that the conclusion in the last sentence is indeed a contradiction. □

9.6 A derived model at κ

Our goal now is to show that if $G \subseteq Coll(\omega, < \kappa)$ is $W[g_0]$-generic and $g_i = G \cap Coll(\omega, < \mu_i)$ then $S_{\mu_i, h_0*g_0*g_i}$ can be dovetailed into one model of determinacy. In the next subsection, we will use this fact to construct a strategy for some $\mathcal{P}_\mu$ which has branch condensation. Our first lemma is a strengthening of Lemma 9.8 and Lemma 9.13.

Lemma 9.36 *Suppose (M, π) is an excellent hull at some excellent μ. Let $g \subseteq Coll(\omega, \mu)$ be W-generic and let $B \subseteq \kappa_M$ be a set in $M[h_0 * g]$. Then $(\mathcal{W}(B))^{M[h_0*g]} = \mathcal{W}(B)$.*

Proof. We clearly have that $(\mathcal{W}(B))^{M[h_0*g]} \trianglelefteq \mathcal{W}(B)$. We need to show that $\mathcal{W}(B) \trianglelefteq (\mathcal{W}(B))^{M[h_0*g]}$. Suppose not. Let then $\mathcal{M} \trianglelefteq \mathcal{W}(B)$ be the least such that $\rho_\omega(\mathcal{M}) = B$ and $(\mathcal{W}(B))^{M[h_0*g]} \triangleleft \mathcal{M}$. To derive contradiction, we need to show that $\mathcal{M} \in M[h_0 * g]$ and $M[h_0 * g] \vDash$ "$\mathcal{M}$ is κ^+-iterable". Let $\mathcal{Q} = \mathcal{P}_\mu^M$. Recall that $\mathcal{Q}$ is short tree iterable in S_{μ, h_0*g}.

Let $C \in M$ be the name for B. We claim that if $\mathcal{T}$ is the correctly guided tree on $\mathcal{Q}$ for making C genericaly generic then $\mathcal{T} \in M$. To see this, it is enough to show that if $\alpha < lh(\mathcal{T})$ is limit then $\mathcal{Q}(\mathcal{T} \restriction \alpha) \in M$. It will then follow that $M \vDash$ "$\mathcal{Q}(\mathcal{T} \restriction \alpha)$ is countably iterable" and hence, M can correctly identify the branch of $\mathcal{T} \restriction \alpha$. Suppose then $\mathcal{Q}(\mathcal{T} \restriction \alpha) \notin M$. Let $\mathcal{S} = (Lp(\mathcal{M}(\mathcal{T} \restriction \alpha)))^M$.

Claim. There is a cofinal wellfounded branch c of $\mathcal{T} \restriction \alpha$ such that $c \in M$.

Proof. Let $\mathcal{T}^* = \pi(\mathcal{T} \restriction \alpha)$ and $\mathcal{S}^* = \pi(\mathcal{S})$. Notice that $\mathrm{cf}^V(\delta(\mathcal{T}^*)) < \kappa$. Let $\lambda = \mathrm{cf}^V(\delta(\mathcal{T}^*))$. Fix an excellent $\eta > \mu$ such that $\eta^\lambda = \eta$. Let $(N, \sigma) > (M, \pi)$ be an excellent hull at η such that $N^\lambda \subseteq N$. Let $\mathcal{U} = \sigma^{-1}(\mathcal{T}^*)$. We have that $\mathcal{U}$ is correctly guided and hence, $\mathcal{U}$ is according to $\pi_{M,N}(\Sigma^M_{\mu,\vec{B}})$. Let then $b = \pi_{M,N}(\Sigma^M_{\mu,\vec{B}})(\mathcal{U})$. Let $\xi \in b$ be least such that $\pi^{\mathcal{U}}_{\xi,b}$ is defined. Because $\mathrm{cf}^V(\delta(\mathcal{U})) = \lambda$, we can fix a cofinal $Y \subseteq rng(\pi^{\mathcal{U}}_{\xi,b})$ of order type λ. We have that $Y \in N$ and b is the unique branch c such that for some $\gamma \in c$, $Y \subseteq rng(\pi^{\mathcal{U}}_c)$. It then follows that $b \in N$. By elementarity, there is a wellfounded branch c of $\mathcal{T} \restriction \alpha$ such that $c \in M$. □

Let then c be as in the claim. If $c = \pi(\Sigma^M_{\mu,\vec{B}})(\mathcal{T} \restriction \alpha)$ then we are done as then $\mathcal{Q}(c, \mathcal{T} \restriction \alpha)$ exists. Suppose then $c \neq \pi(\Sigma^M_{\mu,\vec{B}})(\mathcal{T} \restriction \alpha)$. It then follows that $\mathcal{T} \restriction \alpha$ has two well-founded branches implying that $\mathrm{cf}(\delta(\mathcal{T} \restriction \alpha)) = \omega$. Let $b = \pi(\Sigma^M_{\mu,\vec{B}})(\mathcal{T} \restriction \alpha)$ and let $\xi \in b$ be the least such that $\pi^{\mathcal{T} \restriction \alpha}_{\xi,b}$-exists. Again, by choosing an ω-sequence cofinal in $\delta(\mathcal{T} \restriction \alpha)$ and repeating the proof of the claim using the fact that M is ω-closed, we get that $b \in M$. It then follows that, after all, $\mathcal{Q}(\mathcal{T} \restriction \alpha) \in M$, contradiction!

Let now $\mathcal{T} \in M$ be the correctly guided tree on $\mathcal{Q}$ for making C genericaly generic. Let now $b = \pi(\Sigma^M_{\mu,\vec{B}})(\mathcal{T})$. Repeating the above arguments we can show that $b \in M$. Let then $\mathcal{K} = \mathcal{M}^{\mathcal{T}}_b$. We have that B is generic over $\mathcal{K}$ for the extender algebra of $\mathcal{K}$. Moreover, because $\mathcal{K}$ is full, we have that $\mathcal{W}(B) \in \mathcal{K}[B]$. Hence, $\mathcal{M} \in M$.

To show that $\mathcal{M}$ is κ^+-iterable in $M[h_0 * g]$ we repeat the above argument and the proof of Lemma 9.13. Let Λ be the iteration strategy of $\mathcal{M}$. Given a tree $\mathcal{U} \in M$ on $\mathcal{M}$ according to Λ and of limit length, by the above proof we can find a $\pi(\Sigma^M_{\mu,\vec{B}})$-iterate $\mathcal{S}$ of $\mathcal{Q}$ such that $(\mathcal{M}, \mathcal{U})$ are generic over $\mathcal{S}$. It then follows that $\Lambda(\mathcal{U}) \in \mathcal{S}[\mathcal{M}, \mathcal{U}]$ implying that if $b = \Lambda(\mathcal{U})$ then $b \in M[h_0 * g]$. b can then be identified in $M[h_0 * g]$ as the unique branch of $\mathcal{U}$ such that the phalanx $\Phi(\mathcal{U}^\frown \mathcal{M}^{\mathcal{U}}_b)$ is countably iterable. □

Lemma 9.37 *Suppose (M, π) is a perfect hull at μ_m for some m and let $\mathcal{Q} = \mathcal{P}^M$. Suppose $\sigma : \mathcal{Q} \to_{\Sigma_1} \mathcal{S}$ is such that $\mathcal{S} \in (H_{\kappa_M^+})^M$ and there is $k : \mathcal{S} \to \mathcal{P}$ such that $\pi \restriction \mathcal{Q} = k \circ \sigma$. Then $\mathcal{S} = \mathcal{W}(\mathcal{S}|\delta_\omega^{\mathcal{S}})$.*

Proof. Let $T = T_{\mu_{m+1}}$ (this was defined before Lemma 9.21). Let $B \in \mathcal{P}(\kappa_M) \cap M$ code $\mathcal{Q}, \mathcal{S}$. Let $\eta = \kappa_M$. We then have that $L[T, A_M, B] \vDash H_{\eta^{++}} = \mathcal{W}(\mathcal{W}(A_M, B))$. Let $\xi = \kappa^+$ and let $k : N \to L_\xi[T, A_M, B]$ be an elementary such that $L[T, A_M, B] \vDash$ "$|N| = \eta^+ \wedge crit(k) > \eta^+$" and let $S \in N$ be such that $k(S) = T$. Let $\nu = k^{-1}(\kappa^+)$. Notice that we have that $\mathcal{P}(\delta_\omega^{\mathcal{Q}}) \cap L_\nu[S, \mathcal{Q}] = (\mathcal{P}(\delta_\omega^{\mathcal{Q}}))^{\mathcal{Q}}$. It follows from Lemma 9.36 that $N \in M$

Suppose now that $\mathcal{S} \neq \mathcal{W}(\mathcal{S}|\delta_\omega^{\mathcal{S}})$. Let $\mathcal{M} \trianglelefteq \mathcal{W}(\mathcal{S}|\delta_\omega^{\mathcal{S}})$ be least such that $\mathcal{S} \triangleleft \mathcal{M}$ and $\rho_\omega(\mathcal{M}) = \delta_\omega^{\mathcal{S}}$. Notice that if $g \subseteq Coll(\omega, \mathcal{S})$ is W-generic and $x \in L_\nu[S, \mathcal{S}][g]$ codes $\mathcal{S}$ then there is y coding $\mathcal{M}$ such that $(x, y) \in p[S_n]$ where n is as in Lemma 9.21. This is because there is such a $y \in N[x]$. We can then repeat the proof of Lemma 9.21 to conclude that in fact $\mathcal{M} \trianglelefteq \mathcal{S}$. □

Lemma 9.38 *Suppose $G \subseteq Coll(\omega, < \kappa)$ is $W[g_0]$-generic and let $g_i = G \cap Coll(\omega, < \mu_i)$. Let $\mathbb{R}^* = \cup_{i<\omega}(\mathbb{R}^{W[g_0 * g_i]})$ and let $C_k = \cup_{i<\omega} B_{\mu_i, k, g_i}$. In $W(\mathbb{R}^*)$, let $\Phi = \{D \subseteq \mathbb{R}^* : D$ is Wadge reducible to some $C_k\}$. Then $L(\Phi, \mathbb{R}^*) \vDash AD^+$.*

Proof. It is enough to proof that the claim holds in some perfect (M, π). Fix then such a perfect (M, π) at some μ_m. Let $(N, \sigma) > (M, \pi)$ be a perfect hull at μ_{m+1} and let $\Lambda = \Lambda^{M, N, \mathcal{P}^M}$. Let $N_i = (H_{(\mu_i^M)^+})^M$. Let $\mathcal{Q}$ be the Λ-iterate of $\mathcal{P}^M$ which is obtained via $\langle N_i : i < \omega\rangle$-generic genericity iteration of $\mathcal{P}^M$. We have that if $i : \mathcal{P}^M \to \mathcal{Q}$ is the iteration embedding and $l : \mathcal{Q} \to \mathcal{P}$ is the last move of II then $\pi \restriction \mathcal{P}^M = l \circ i$ implying that $\mathcal{Q} = \mathcal{W}(\mathcal{Q}|\delta_\omega^{\mathcal{Q}})$ (this follows from Lemma 9.37). Fix now some $G \subseteq Coll(\omega, < \kappa_M)$ generic over $M[h_0 * g_0]$ and let g_i's be defined similarly as above. We define C_k^M and Φ^M similarly working in $M[h_0 * g_0 * G]$. Let $\tau_k = \oplus_{i<\omega} \tau_{B_{\mu_m, k, G}}^{\mathcal{Q}|((\delta_i^{\mathcal{Q}})^{+\omega})^{\mathcal{Q}}}$. We have that $\tau_k \in \mathcal{Q}$ and therefore, letting $\mathbb{R}^* = \cup_{i<\omega}(\mathbb{R}^{M[g_i]})$, we have that $B_{\mu_m, k, G} \cap \mathcal{Q}(\mathbb{R}^*) = C_k^M$. Notice that in $\mathcal{Q}(\mathbb{R}^*)$, $\Phi^M \subseteq \{A \subseteq \mathbb{R}^* : L(A, \mathbb{R}^*) \vDash AD^+\}$. It then follows from the derived model theorem that $L(\Phi^M, \mathbb{R}^*) \vDash AD^+$. □

Given $G \subseteq Coll(\omega, < \kappa)$, we let $\mathcal{S}_G = L(\Phi, \mathbb{R}^*)$. Similarly, if (M, π) is a perfect hull at some μ_p and $G \subseteq Coll(\omega, < \kappa_M)$ is M-generic then we let $\mathcal{S}_G^M = (L(\Phi, \mathbb{R}^*))^{M[G]}$.

9.7 Strongly $\vec{B}$-guided strategy

Lemma 9.39 *Suppose (R,τ) is an excellent hull at μ_0 and let $\mathcal{R} = \mathcal{P}^R$. Suppose $g \subseteq Coll(\omega,\mu_0)$ is generic and $p<\omega$. Then some tail of $\Sigma^R_{\mu_0,\vec{B}}$ strongly respects B_p.*

Proof. Suppose not. Let $\Psi = \Sigma^R_{\mu_0,\vec{B}}$. Fix $\langle \mathcal{R}_j, \mathcal{Q}_j, \sigma_j, \nu_j, \vec{\mathcal{T}}_j : j<\omega\rangle \in M$ such that

1. $\mathcal{R}_0 = \mathcal{R}$,
2. $\vec{\mathcal{T}}_j$ is a stack according to Ψ on $\mathcal{R}_j$, $\mathcal{R}_{j+1}$ is its last model and $\pi^{\vec{\mathcal{T}}_j}$-exists,
3. $\sigma_j : \mathcal{Q}_j \to \mathcal{R}_{j+1}$ and $\nu_j : \mathcal{R}_j \to \mathcal{Q}_j$ are elementary
4. $\pi^{\vec{\mathcal{T}}_j} = \sigma_j \circ \nu_j$ and
5. $\sigma_j^{-1}(\tau^{\mathcal{R}_{j+1}}_{B_p}) \neq \tau^{\mathcal{Q}_j}_{B_p}$.

We let $\mathcal{R}_\omega$ be the direct limit of $\mathcal{R}_j$'s under $\pi^{\vec{\mathcal{T}}_j}$'s. We have that there is $l : \mathcal{R}_\omega \to \mathcal{P}_\omega$. Fix some perfect hull $(M,\pi) > (R,\tau)$ at μ_1 and let $\Sigma = \Lambda^{R,M,\mathcal{R}_\omega}$. Let $\pi_{i,\omega} : \mathcal{R}_j \to \mathcal{R}_\omega$ be the direct limit embedding and let $\sigma_{j,\omega} = \pi_{j+1,\omega} \circ \sigma_j$.

Next let $N_i = (H_{(\mu_i^R)^+})^R$ and let $h \subseteq Coll(\omega,\kappa_R)$ be R-generic such that $h \in W[g]$. Let $\sigma = \cup_{i<\omega}(\mathbb{R}^{N_i[h\cap Coll(\omega,\mu_i^R)]})$. Let $\langle \mathcal{R}^j_k, \mathcal{Q}^j_k, \vec{\mathcal{S}}^j_k, \vec{\mathcal{W}}^j_k, \sigma^j_k, \nu^j_k, m^j_k : j,k<\omega\rangle$ be the iteration coming from simultaneous $\langle N_i : i<\omega\rangle$-generic genericity iteration of $\langle \mathcal{R}_i, \mathcal{Q}_i : i<\omega\rangle$ via Σ. Also, we let $\langle \mathcal{R}^\omega_k, \mathcal{Q}^\omega_k, \vec{\mathcal{S}}^\omega_k, \vec{\mathcal{W}}^\omega_k, m^\omega_k, \sigma^\omega_k, \nu^\omega_k : k<\omega\rangle$ be the direct limit of $\langle \mathcal{R}^j_k, \mathcal{Q}^j_k, \vec{\mathcal{S}}^j_k, \vec{\mathcal{W}}^j_k, m^j_k, \sigma^j_k, \nu^j_k : j,k<\omega\rangle$.

It is not difficult to see that σ is indeed the set of reals of some symmetric generic extension of each $\mathcal{R}^\omega_k$ and $\mathcal{Q}^\omega_k$. Moreover, letting D_k and C_k be the derived models of respectively $\mathcal{R}^\omega_k(\sigma)$ and $\mathcal{Q}^\omega_k(\sigma)$ we have that for every k,

$$\mathcal{S}^R_h \subseteq D_k \cap C_k.$$

Notice that $B_p \cap \sigma \in \mathcal{S}^R_h$. Let then $s \in Ord^{<\omega}$ be such that $B_p \cap \sigma$ is definable from s in $\mathcal{S}^R_h$. Let $\xi = \Theta^{\mathcal{S}^R_h}$. Notice then for each k, in D_k, $B_p \cap \sigma$ is a set definable from s in $L(P_\xi(\mathbb{R}))$. Let then k be such that for all $n>k$,

$$m^\omega_n(s) = s \text{ and } m^\omega_n(\xi) = \xi.$$

We then have that for every $n<\omega$,

$$\sigma_n^\omega(s,\xi) = \nu_n^\omega(s,\xi) = (s,\xi).$$

But this implies that for $n > k$, $\sigma_n^\omega(\tau_{B_p}^{\mathcal{Q}_n^\omega}) = \tau_{B_p}^{\mathcal{R}_{n+1}^\omega}$. This then implies that for every $n > k$, $\sigma_n(\tau_{B_p}^{\mathcal{Q}_n}) = \tau_{B_p}^{\mathcal{R}_{n+1}}$, contradiction. □

Applying Lemma 9.39 repetitively and using the proof of Lemma 6.6, we get that

Corollary 9.40 *Suppose (R,τ) is an excellent hull at μ_0 and let $\mathcal{R} = \mathcal{P}_{\mu_0}^R$. Suppose $g \subseteq Coll(\omega,\mu_0)$ is generic. Then some tail of $\Sigma_{\mu_0,\vec{B}}^R$ has branch condensation.*

9.8 Finishing the proof of Theorem 0.1

Fix then (R,τ) as in the corollary above. It then follows that some tail of $\Sigma =_{def} \tau(\Sigma_{\mu_0,\vec{B}}^R)$ has branch condensation. Let $\mathcal{Q}$ be the Σ-iterate of $\mathcal{P}_{\mu_0}$ such that letting Λ be the corresponding tail of Σ, Λ has a branch condensation. We then have that Λ is fullness preserving, it has branch condensation and it is a (κ,κ)-iteration strategy. Appealing to Lemma 9.9, we can assume Λ is a (κ^+,κ^+)-strategy. Let μ be an excellent point such that $\mathcal{Q} \in H_\mu$. It then follows from Lemma 9.3 that if $g \subseteq Coll(\omega,\mu)$ is W-generic then in $W[g]$, $L^\Lambda(\mathbb{R}) \vDash AD^+$. It then also follows that in $W[g]$, since Λ is a fullness preserving iteration startegy and that $L^\Lambda(\mathbb{R}) \vDash MC$, $L^\Lambda(\mathbb{R}) \vDash AD^+ + \theta_0 < \Theta$, contradiction! See Remark 4.2 for an explanation that the conclusion in the last sentence is indeed a contradiction.

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