Connected Groups of Finite Morley Rank

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Themes

I Connected groups of finite Morley rank (in general)

II Generic covering and conjugacy theorems

III Semisimple torsion

Essential Notions—Generalities

- Morley rank $(\mathrm{rk}\,(\mathrm{X}))$
- Generic set: $\mathrm{rk}\,(\mathrm{X}) = \mathrm{rk}\,(\mathrm{G})$
- Connected group

$$[G : H] < \infty \implies G = H.$$

$$X, Y \subseteq G \text{ generic} \implies X \cap Y \text{ generic}$$

- $d(X)$: definable subgroup generated by X.
- Fubini: Lascar-Borovik-Poizat

Essential Notions—*p*-groups and Types

- *p*-torus: divisible abelian *p*-group
- Types:

  Degenerate: No infinite 2-subgroup
  Even: Nondegenerate, no nontrivial 2-torus
  (“characteristic two type”)
- *p*-unipotent: definable, connected, bounded exponent, nilpotent *p*-group

The Algebraicity Conjecture

Conjecture (Algebraicity)

G: finite Morley rank, connected.
H: maximal connected solvable normal, definable.

$$1 \to H \to G \to \bar{G} \to 1$$

$\bar{G}$: a central product of algebraic groups.

Equivalently: The simple groups are algebraic.

Borovik Programme

- FSG (15,000 pp., or 5,000 pp.)
- (No bad fields)
- Minimal Counterexample

. . . The perils of incomplete inductive arguments . . .

Groups without 2-tori (I)

$$1 \to O_2(G) \to G \to \bar{G} \to 1$$

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Definition

$U_2(G) = \langle U \leq G : \text{2-unipotent}\rangle$.

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Ingredients

Theorem (E,M)

A simple group of even type is algebraic.
There are no simple groups of finite Morley rank of mixed type.

Theorem (D)

A connected degenerate type group contains no elements of order two.

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Methods: Black box group theory, genericity arguments—soft methods

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From FSG to Geometry (good tori). (More below.)

Groups with 2-Tori

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Back to the Borovik Programme: bounds on Prüfer 2-rank.

Theorem (Borovik, Burdges, Cherlin, Jaligot)

In a minimal connected nonalgebraic simple group of finite Morley rank, the Prüfer 2-rank is at most 2.

—Burdges unipotence theory for elimination of hypotheses on bad fields.
—Analysis of minimal simple groups: Deloro (with technology of Burdges, Frécon).

Groups without 2-unipotent subgroups

In a more geometrical vein ...

Theorem

- *2-elements are toral.*
- *Maximal 2-tori are conjugate.*
- *Any 2-element in the centralizer of a maximal 2-torus belongs to that 2-torus.*
- *The generic element of G belongs to $C^\circ(T)$ for a unique maximal 2-torus T.*

But this is a shift in emphasis ...

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Rigidity properties:

- R-I $N^\circ(T) = C^\circ(T)$
- R-II Any uniformly definable family of subgroups of T is finite.

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Theorem

- *The multiplicative group of a field of finite Morley rank is a good torus [Wagner].*
- *Maximal good tori are conjugate [Cherlin].*

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Limoncello (Even type with strongly embedded subgroups IV):
finiteness of the number of conjugacy classes of 1-dimensional algebraic tori contained in a fixed definable subgroup.

Generic Covering and Conjugacy

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Properties of $H = C^\circ(T)$:

- Almost self-normalizing (Rigidity-I)
- Generically disjoint from its conjugates: $H \setminus (\bigcup H^{[G \setminus N(H)]})$ generic in H.

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Lemma (Genericity Lemma)

If a definable subgroup H of G is almost self-normalizing and generically disjoint from its conjugates then:

- *$\bigcup H^G$ is generic in G;*
- *For $X \subseteq H$, we have $\bigcup X^G$ generic in G if and only if $\bigcup X^H$ is generic in H.*

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Carter Subgroups

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Theorem (Frécon)

In a K^*-group, Carter subgroups are conjugate.

A tour de force. This is a case where a minimal counterexample eventually dies completely. Along the way, Burdges' Bender method is used, and many other things.

Permutation Groups

Theorem (BC)

(G, X) definably primitive. Then $rk\,(G)$ is bounded by a function of $rk\,(X)$.

Definably primitive: no nontrivial G-invariant definable equivalence relation.

MPOSA = Macpherson-Pillay/O'Nan-Scott-Aschbacher
A description of the socle of a primitive permutation group, and the stabilizer of a point in that socle.

- Affine: The socle A is abelian and can be identified with the set X on which G acts.
- Non-affine: The socle is a product of copies of one simple group.

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Proposition

(G, X) definably primitive. Then the degree of multiple transitivity of G is bounded by a function of rk (X).

(Special case of the theorem, but sufficient.)

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The upper bound: $rk(T/O_\infty(T)) \leq rk(X)$. This is because the stabilizer of a generic element of X is torsion-free.

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The upper bound: $rk\,(T/O_\infty(T)) \leq rk\,(X)$. This is because the stabilizer of a generic element of X is torsion-free.

But the lower bound requires attention.

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The group G will induce the action of Sym_t on any t independent generic points.

Trading T in for a smaller torus, and trading t in for a smaller value as well (but not too small) we can set this up so that we have:

- a finite group Σ operating on T, and
- covering Sym_t, and
- sitting inside a connected group H such that
- T is the definable hull of a maximal 2-torus in H.

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It then seems reasonable that this action can be exploited to blow up T, and also T/T_∞.
There is a glaring hole in this argument.

Plugging a hole

The Setup

T inside G, G connected, Sym_t acts on T, t large, and T is the definable hull of a maximal 2-torus.

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However: at this point G can again be taken to be simple (via MPOSA) and therefore a dichotomy applies:

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- G contains no unipotent 2-subgroup.

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And so, we are done!

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- The original goal: a list of “intractable” minimal configurations.
  I would like to see that in final form!
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  Popov in characteristic 0*, with algebraic actions.*
- L-group theory for odd type groups.
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- And a pony!