Permutation Groups of Finite Morley Rank

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Reduction of a theorem of permutation group theory to the simple case, and solution by “structure theory.”

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Theorem (with Borovik)

The rank of G can be bounded in terms of the rank of X, if the action of G is faithful and definably primitive.

Method: structure theory for *simple* groups of finite Morley rank.

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As it happens, these notions are equivalent except when the point stabilizer G_α is finite.

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Definably primitive: no nontrivial definable invariant equivalence relation on X.

Group-theoretically: $G_\alpha < G$ maximal, or definably maximal.

Examples

Intransitive [Gropp]

Ingredients: field K, vector space V, 1-dim space L, and maps

$$\lambda : \mathrm{V} \twoheadrightarrow \mathrm{L} \text{ linear; } \mathrm{f} : \mathrm{L} \to \mathrm{V} \text{ space curve}$$

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Faithful action on a rank 2 set of unbounded rank.
Orbits: $L_x = \{x\} \times L$. Kernel of the induced action:

$$A.v \in \ker \lambda \qquad (v = f(x))$$

Varying with x.

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Imprimitive

G algebraic acting on V, $W \leq V$ containing no G-invariant subspace. $\hat{G} = V \rtimes G$ acting on the coset space $W \backslash \hat{G}$.
We want: G fixed, dim(V/W) fixed, dim$(V) \to \infty$.

- G simple, V irreducible, W a hyperplane.
- G a torus, all eigenspaces 1-dimensional, W avoids them.

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Generically t-transitive: generically transitive on X^t.

Not fully classified, even for actions of algebraic groups (Popov).

Bounds

(G, X) finite Morley rank, $r = \mathrm{rk}(X)$ (fixed)

$\rho(r)$: sup($\mathrm{rk}(G)$) with (G, X) definably primitive.

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$\rho(r)$: sup$(\mathrm{rk}\,(G))$ with (G, X) definably primitive.
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$\tau(r)$: sup $t(G, X)$ i.e. the degree of generic multiple transitivity, with (G, X) definably primitive.
$\rho_S(r)$: sup(rk (G)) with r fixed, G simple, acting faithfully on X.
$\tau_S(r)$: sup $t(G, X)$ i.e. the degree of generic multiple transitivity, with (G, X) definably primitive and simple.

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Theorem

All of these are bounded in terms of $\mathrm{rk}(X)$.

Bounds

Ranks of:

- Simple groups acting transitively;
- Generically highly transitive definably primitive (or simple) groups;
- Definably primitive groups.

Theorem

All of these are bounded in terms of $\mathrm{rk}\,(\mathrm{X})$.

Strategy: Definably primitive $\rightarrow$ generically highly transitive $\rightarrow$ simple transitive $\rightarrow$ simple highly transitive $\rightarrow$ structure theory.

From τ to ρ or τ_S to ρ_S

ρ: $\mathrm{rk}\,(G)$; τ: degree of generic transitivity

Proposition

$r \cdot \tau(r) \leq \rho(r) \leq r\tau(r) + \binom{r}{2}$.

Remark: For transitive groups $\mathrm{rk}\,(G) \geq \mathrm{rk}\,(X)$ and so $\rho \geq r \cdot \tau$.

From τ to ρ (cont.)

o_k: the *generic rank* of an orbit with k-independent elements fixed . . .

$\alpha = (\alpha_1, \ldots, \alpha_k)$ generic and independent

$$\{x \in X : rk\,(x^{G_\alpha{}^\circ}) = o_k\} \text{ generic in } X$$

From τ to ρ (cont.)

o_k: the *generic rank* of an orbit with k-independent elements fixed ...

Lemma

If $0 < o_k < rk(X)$ *then* $o_{k+1} < o_k$.

In short order $o_k = 0$, $G_\alpha{}^\circ$ acts generically trivially, and G_α is finite.

From τ to ρ (cont.)

o_k: the *generic rank* of an orbit with k-independent elements fixed ...

Lemma

If $0 < o_k < rk(X)$ *then* $o_{k+1} < o_k$.

Idea: $o_{k+1} = o_k$ means orbits are generically unaffected by fixing an independent point; this reveals a G-invariant definable equivalence relation, whose classes are approximately these orbits.

MP-OSA

G generically highly transitive and definably primitive. The objective is to reduce to the simple case.
Finite group theory: O'Nan-Scott-Aschbacher (OSA)
Finite Morley rank: Macpherson-Pillay (MP-OSA)
Definable Socle: the group generated by minimal definable normal subgroups.

MP-OSA

G generically highly transitive and definably primitive. The objective is to reduce to the simple case.

Theorem (MP-OSA, preamble)

(G, X) definably primitive with socle B. Two cases:

- *Affine case: B abelian, $G = B \rtimes G_\alpha$. The action of G on X corresponds to the action of G on B by translation and conjugation. B is G_α-minimal.*
- *Else $B = T_1 \times \cdots T_k$ is a product of isomorphic simple groups.*

Simple groups are at least in view here. To bring them onto the scene as the central characters takes some more analysis.

Affine groups

$G = A \rtimes H$ (H acts on A).
Use $rk\,(A)$ to control $rk\,(H)$, (A H-minimal, i.e. G definably primitive).

Theorem

Let (H, A) be as stated and $r = rk\,(A)$. Then one of the following holds.

- *A is torsion free, divisible: then $rk\,(H) \leq r^2$*
- *A is elementary abelian: in this case, if every definable simple nonabelian subgroup of H has rank at most s, then*

$$rk\,(H) \leq \max(r^2, r(s^2 + s))$$

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The bound r^2 corresponds to the linear case: r being the "dimension" of the "space" on which H acts. The proof reflects this idea.

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- *...*

If A is *torsion free* then A is pointwise definable from a sequence of at most r elements $a = (a_1, \ldots, a_n)$ and therefore $G_a = 1$, $rk\,(G) \leq nr \leq r^2$.

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The case *A elementary abelian* takes more machinery, notably Wagner's results on fields of finite Morley rank.

Nonabelian socles

(G, X) definably primitive with nonabelian definable socle, and $r = rk\,(X)$.

Lemma

If G has more than one minimal normal definable subgroup, then $rk\,(G) \leq r^2 + 2r$.

(Quite similar to the case of divisible abelian socle in fact.)

More MP-OSA

The other case:

Notation

L is the unique minimal normal definable subgroup of G;
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There are two possibilities:

(a) The point stabilizer is $L_\alpha = \prod_i (L_i)_\alpha$;
(b) $L = \prod L_I$ with each L_I a product of k simple factors L_i ($i \in I$), $k \geq 2$, and $(L_I)_\alpha$ a diagonal subgroup.

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Lemma

Correspondingly, with $s = rk\,(L_i)$, we have

(a) $rk\,(G) \leq r(s + s^2)$;

(b) $rk\,(G) \leq 2(r^2 + r)$.

What is wanted is a bound on s.

Involutions

A generically t-transitive group, with $t \geq 2$ contains an element of order 2.

Theorem (Classification)

Let G be a simple group of finite Morley rank containing an involution. Then one of the following holds.

- *G is a simple algebraic group.*
- *G contains a nontrivial 2-torus.*

2-*torus:* Divisible abelian 2-group.
So much for structure theory.

Involutions

Theorem (Classification)

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The rest of our strategy comes down to *control of p-tori*. Objectives:

- Bound the ranks of p-tori above in terms of r;
- Bound the ranks of p-tori below in terms of t.

Actions of *p*-tori

Lemma

Let (G, Ω) be a definably primitive permutation group of finite Morley rank, T a definable divisible abelian abelian subgroup, and $O(T)$ its largest definable torsion free subgroup. Then

$$rk\,(T/O(T)) \leq rk\,(X)$$

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Proof.

T_0 be the torsion subgroup of T. Let $\alpha \in X$ be generic over T_0. Then $T_\alpha \cap T_0 = 1$.

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Proof.

T_0 be the torsion subgroup of T. Let $\alpha \in X$ be generic over T_0.
Then $T_\alpha \cap T_0 = 1$.
So $T_\alpha \leq O(T)$ and

$$rk\,(T/O(T)) \leq rk\,(T/T_\alpha) = rk\,(\alpha^T) \leq rk\,(X)$$

□

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In particular if G is algebraic and T is a maximal torus, its algebraic dimension is bounded by $rk\,(X)$, after which the classification of simple algebraic groups suffices.

Sporadic groups?

Key case: a generically highly transitive simple group G with a nontrivial 2-torus of bounded Morley rank.
So what?

Sporadic groups?

Key case: a generically highly transitive simple group *G* with a nontrivial 2-torus of bounded Morley rank.
So what?

If *G* is generically *t*-transitive and $\alpha = (x_1, \ldots, x_t)$ is a sequence of generic and independent elements of *X*, then *G* induces the action of Sym_t on this set.

Sporadic groups?

Key case: a generically highly transitive simple group G with a nontrivial 2-torus of bounded Morley rank.
So what?

Better:
If G is generically $2t$-transitive then G_α is generically t-transitive and $N(G_\alpha)$ induces the action of Sym_t on these elements.

Sporadic groups?

Key case: a generically highly transitive simple group G with a nontrivial 2-torus of bounded Morley rank.
So what?

Simplify: imagine Sym_t acting on G_α; even better, on a maximal 2-torus T of G_α. From there one would expect to *pump up* the rank of T, so bounding the rank of T above would finally control t!

A little too enthusiastic ...?

A maximal 2-torus

(G, X) highly generically transitive.
Imagine: H a connected definable subgroup, also pretty highly generically transitive.

$$\alpha = (x_1, \ldots\ldots, x_n)$$

a *long* sequence of generic independent points. T a maximal 2-torus of H_α. And Σ, some finite group resembling Sym_n, acting on T.

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We have a bound on $rk\,(T/O(T))$, namely $rk\,(X)$.
If Σ acts faithfully, then yes, we can pump up $rk\,(T/O(T))$ in terms of n i.e. bound n in terms of $rk\,(X)$—and be done.

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Let H be simple of finite Morley rank, T a maximal 2-torus (nontrivial). Then T contains all the 2-elements in $C(T)$.

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Lemma

(G, X) simple, generically highly transitive, $r = rk(X)$. Then for any k there is a connected subgroup H of G with

$$rk(G) - rk(H) \leq (kr + 1)r$$

so that the point stabilizer H_α contains a maximal 2-torus of H (α an independent generic sequence of length k).

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These two together give a *faithful* action of “Sym_k”.

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so that the point stabilizer H_α contains a maximal 2-torus of H (α an independent generic sequence of length k).

So $rk\,(T/O(T))$ grows with t (linearly) and is bounded by r.
(Done)

Many problems

The bounds are weak.

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$r = 1$: $\rho = 3$, $\tau = 3$ (Hrushovski)

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$r = 2$: $\tau \leq 27$. Gropp: sharp generic transitivity ≤ 6.

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$r = 1$: $\rho = 3$, $\tau = 3$ (Hrushovski)
$r = 2$: $\tau \leq 27$. Gropp: sharp generic transitivity ≤ 6.

Problem (Borovik)

Let G be connected, acting transitively and generically $(n+2)$-transitively on a set of Morley rank n. Then G is the projective group acting on projective space over an algebraically closed field.

(For 18 more or less related problems, see *Permutation groups of finite Morley rank*, Newton Proceedings Volume, to appear.)