Between model theory and combinatorics: Homogeneity, WQO, Universality

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THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

Colloque en l'honneur de Chantal Berline
June 4
(9:30–10:25)

Between Model Theory and Combinatorics

I Homogeneous Structures
- Distance Homogeneous Graphs

II Universal Graphs
- Trees

III Well quasi-orders
- Finiteness Theorem

Homogeneity

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Urysohn 1927 (Ph.D. 1921; d. 1924, aged 26): $\mathbb{U}$

Rado 1964: G_∞

Berline-Cherlin 1980-1983: QE rings
(cf. Boffa/Macintyre/Point, Baldwin/Rose, Saracino/Wood)

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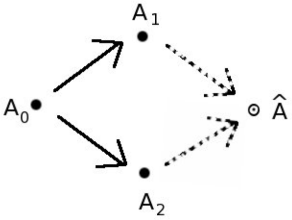

Amalgamation of Metric Spaces

1-point extensions: $A_i = A_0 \cup \{u_i\}$.

$d^+(u_1, u_2) = \min(d(u_1, a) + d(u_2, a))$
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$\mathbb{U}_0$: The universal homogeneous countable rational-valued metric space.
$\mathbb{U}$: The completion of $\mathbb{U}_0$.

Homogeneous Graphs and Digraphs

Henson 1971: G_n (K_n-free graph), its automorphisms and structure
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Tools: Fraïssé, Finite Ramsey theorem

Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Color classes $c_1 \le c_2 \le c_1$*, densely colored; connections between pairs of color class components; triples.* Fraïssé for existence.

Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Kechris-Pestov-Todorcevic 2005:
Fraïssé+Ramsey+Top. Dynamics

Glasner: “This remarkable paper is a tour de force where three experts in disparate areas—model theory, structural Ramsey theory and topological dynamics—collaborate in creating a unified and beautiful theory.”

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Minimal flows: compact actions with every orbit dense.
Extremely amenable: no nontrivial minimal flow

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• The extremely amenable closed subgroups of Sym_∞ are exactly the groups of the form Aut($\mathbb{C}$) with $\mathbb{C}$ the Fraïssé limit of a Fraïssé order class with the Ramsey property.
• If $\mathbb{C}$ is one of the following structures, then the universal minimal flow $\mathrm{M(G)}$ of the group $\mathrm{G} =$ Aut($\mathbb{C}$) is its action on the space of linear orderings of the universe of $\mathbb{C}_0$:

- $\mathrm{G_n}$ $(\mathrm{n} \le \infty)$;
- $(\mathbb{N}, =)$;
- $\mathbb{U}_0$

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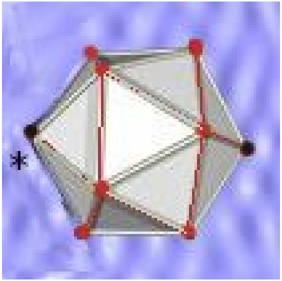

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3. Fraïssé type
   - $\delta \leq d$;
   - Omit $(1, d)$-subspaces ($d \geq 3$);
   - Omit odd cycles up to order $2K + 1$;
   - Omit triangles of perimeter $\geq C$.

     Some interactions in these constraints.

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But (Γ_{k-1}, Γ_k) should be.
Extend the classification project?

Universal Graphs

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Data: Finitely many constraints $\mathcal{C}$ (finite, connected "forbidden" graphs).
Universal countable $\mathcal{C}$-free graph?

? Decidable ?

Universality and $\aleph_0$-categoricity

Existentially complete $\mathcal{C}$-free graphs.
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If the existentially complete countable graph is unique, then it is universal.
And there is an exact criterion for this in terms of the *algebraic closure*.

L’interdit (Givenchy, 1957)

Our forbidden structures are forbidden in the graph theorist’s sense, not the model theorist’s (“induced”) sense.

N.B.: if one takes induced substructures then one gets domino problems if the language is rich enough (maybe not in graphs??)

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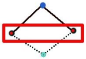
• Forbid C_4. Then for points u, v at distance 2, the "midpoint" is a definable function $f(u, v)$. Such points are in the "definable closure" of u, v.



$\neg C_4$

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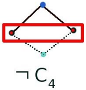

• Forbid a star S_k. Then for any u, the neighbors of u are "algebraic" over u: they lie in a u-definable finite set. (So the algebraic closure of a point is its connected component.)

$\aleph_0$-categoricity and algebraic closure

Theorem (CSS 1999)

Let $\mathcal{C}$ be a finite set of forbidden graphs, T the theory of the existentially complete $\mathcal{C}$-free graphs. Then the following are equivalent.

1. *T has a unique countable model*
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Proof.

$\Longrightarrow$: General nonsense (Ryll-Nardzewski, Engeler, Svenonius)

$\Longleftarrow$: Close analysis: over any finite algebraically closed set, the set of types is finite. □

Applications: Cycles . . .

Conjectured by Menachem Kojman:

Theorem

If $\mathcal{C}$ is closed under homomorphism (i.e., the image of a constraint in $\mathcal{C}$ under graph homomorphism is $\mathcal{C}$-forbidden) then acl is degenerate and there is a universal $\mathcal{C}$-free graph.

Example. Odd cycles.

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Theorem (Cherlin-Shi 1996)

For $\mathcal{C}$ a finite set of cycles the following are equivalent.

1. *There is a universal $\mathcal{C}$-free graph.*
2. *$\mathcal{C}$ consists of all odd cycles up to a fixed length.*

. . . and trees

Theorem (Cherlin-Shelah 2007)

For $\mathcal{C} = \{T\}$ a single tree, the following are equivalent.

1. *There is a universal $\mathcal{C}$-free graph.*
2. *The tree T is an extension of a path by at most one additional edge.*

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$\Longrightarrow$:

Shelah's idea: Pruning

To prune a tree T: T' is obtained by removing all leaves.

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Lemma

If there is a T-free universal graph G then there is a T'-universal graph G^*, consisting of the vertices of G of infinite degree.

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Minimal trees: those which prune to a path or near-path.

(15 cases).

General Pruning

In general: Remove a minimal block-leaf. (Or a downward-closed family.)

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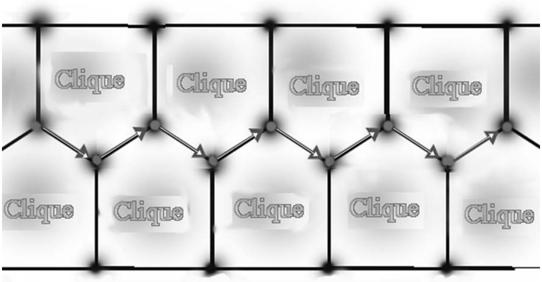
For a single connected constraint C, the problem of determining whether there is a universal C-free graph is algorithmically decidable.

A Concrete Example

The Bouquet $K_5 \wedge K_5$

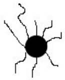


$(K_5 + K_5)$ - free

(Algebraic closure running along the mid-line)

The Hairy Ball Graph

- The Hairy Ball Problem Let K be a finite graph consisting of a complete graph together with a single finite path attached to each vertex. Is there a universal K-free graph?



Equivalently: if one strings together an infinite series of "canonical obstructions" (K minus part of one path) along a 2-way infinite path, does the graph K necessarily appear?

WQO

Well-founded: no descending chains.
WQO: no descending chains or infinite antichains.

Classes of finite structures ordered by embedding (in either of the two common senses) are well-founded, but not in general WQO.

Robertson-Seymour: Finite graphs under "graph minor" are WQO.

Friedman: this is (formally speaking) not easy to prove—that is, it requires impredicative methods.

A dichotomy?

$\mathcal{Q}$: our favorite quasi-order (e.g., all finite tournaments)
$C \subseteq \mathcal{Q}$ finite (constraints, “forbidden” points)
$\mathcal{Q}_C$: C-free elements—$\neg\exists c \in C(x \geq c)$

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Meta-Problem: Can you tell?

Thesis: This is a dichotomy only if one can decide algorithmically which case one is in.

An Example (Friedman)

For L a linear order, let L_{WO} be the largest initial segment of L which is well ordered.

Let $<_1$ be a recursive ordering of $\mathbb{N}$ so that $(\mathbb{N}, <)_{WO}$ is complete Π^1_1.

Let $\mathcal{Q}^*$ be the quasiorder of $\mathbb{N}$ defined by

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Then:

- $\mathcal{Q}^*$ is well-founded;
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Corollary

In the effectively given quasiorder $\mathcal{Q}^*$, recognizing those constraints c which correspond to wqo ideals is as difficult as it could possibly be.

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Paths with colored vertices: 

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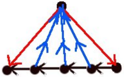
Tournaments: 

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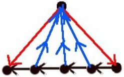
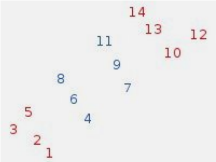
Tournaments: 

Permutations: 

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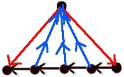
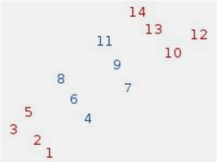
Tournaments:  Permutations: 

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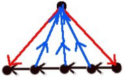
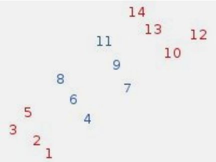
Tournaments: 

Permutations: 

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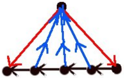
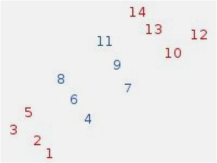
Tournaments:  Permutations: 

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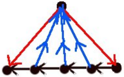
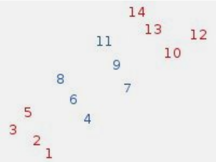
Tournaments:  Permutations: 

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- Colored Paths

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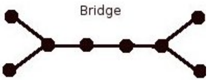
  I_1: Bridges:  (not isolated)

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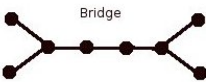
  I_1: Bridges: 

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Proposition

Among vertex-colored paths, the minimal antichains are quasi-periodic, that is they consist of a periodic part augmented by a first and last vertex which break the periodicity.

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Corollary

In the cases of graphs and colored paths, the isolated minimal antichains are dense, the associated ideals are effectively recognizable, and the recognition of wqo classes given by finitely many constraints is effective, in polynomial time.

A finiteness Theorem

Theorem (Cherlin-Latka 2000)

Let $\mathcal{Q}$ be a wellfounded quasiorder. Then for each k, there is a finite set Λ_k of minimal antichains, such that any non-wqo $\mathcal{Q}_{\mathcal{C}}$ with $|\mathcal{C}| \leq k$ allows one of the antichains in Λ_k (up to a finite set).

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Proof.

Induction. Start with $\Lambda_{k+1} = \Lambda_k$ and consider constraints $\mathcal{C} = \{c_1, \ldots, c_{k+1}\}$ for which this is inadequate.

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$\mathcal{C} \in \prod_i \mathcal{Q}^{<I_i}$ a wqo. So there are finitely many minimal cases; expand Λ_k by witnesses for the minimal cases. □

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Corollary

If the ideals $\mathcal{Q}^{<I}$ are computable for $I \in \Lambda_k$, then the decision problem for wqo with respect to $k+1$ constraints is decidable.

Friedman again

Claims:

1. The finiteness theorem for $k = 1$ is provably equivalent to $\Pi^1 1_1 - CA_0$ over RCA_0, even for locally finite quasiorders.
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At what the other extreme we may conjecture:

Conjecture

The isolated minimal antichains are dense for $\mathcal{Q}$ the quasiorder of tournaments, and the corresponding ideals are uniformly recursive.

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. . . and best wishes to Chantal Berline, and for the fruitful interaction of model theory, combinatorics, and computer science . . .