Torsion in Groups of Finite Morley Rank

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Zilber Geometric Model Theory Conference
March 25-28

I Connected Groups
- Structure

II Permutation Groups
- Bounds on Rank

III Torsion
- Centralizers
- Semisimplicity
- Sylow Theorem
- Weyl Group

Essential Notions—Generalities

- Morley rank $(\mathrm{rk}\,(X))$
- Connected group

  $[G : H] < \infty \implies G = H.$

  $X, Y \subseteq G$ generic $\implies X \cap Y$ generic
- $d(X)$: definable subgroup generated by X.
- Fubini: Zilber-Lascar-Borovik-Poizat

The Algebraicity Conjecture

Conjecture (Algebraicity)

G: finite Morley rank, connected.
H: maximal connected solvable normal, definable.

$$1 \to H \to G \to \bar{G} \to 1$$

$\bar{G}$: a central product of algebraic groups.

Equivalently: The simple groups are algebraic.

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Theorem (ABC, 2008)

$$1 \to U_2(G) \to G \to \bar{G} \to 1$$

$U_2(G)$: $1 \to O_2(G) \to \prod_i L_i$ (char 2, Altınel's Jugendtraum - and his habilitation - and Wagner's good tori)
$\bar{G}$: Connected 2-Sylow divisible abelian. ("odd type")

Odd Type: Torsion

Theorem (Degenerate Type)

If there is no nontrivial connected abelian p-subgroup, then there is no p-torsion.

Theorem (Burdges-Altınel)

The centralizer of a divisible torsion subgroup is connected.

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If there are no p-unipotent subgroups, then any p-element which centralizes a maximal divisible p-subgroup T lies in T.

Proof.

T the definable hull of a maximal divisible p-subgroup.
$H = C(T)/T$ connected.
H has no p-torsion. □

MPOSA

Definably primitive: no nontrivial G-invariant definable equivalence relation.

Theorem (BC)

(G, X) definably primitive. Then $rk\,(G)$ is bounded by a function of $rk\,(X)$.

MPOSA = Macpherson-Pillay/O'Nan-Scott-Aschbacher
A description of the socle of a primitive permutation group, and the stabilizer of a point in that socle.

- Affine: The socle A is abelian and can be identified with the set X on which G acts.
- Non-affine: The socle is a product of copies of one simple group.

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Proposition

(G, X) definably primitive. Then the degree of multiple transitivity of G is bounded by a function of $\mathrm{rk}\,(X)$.

(Special case of the theorem, but sufficient.)

Bounds on t

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Strategy: Let T be a maximal 2-torus.

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But the lower bound requires attention.

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We want to show that a large degree of generic transitivity (t large) blows up $rk\,(T/T_\infty)$ for T the definable hull of a 2-torus.

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The group G will induce the action of Sym_t on any t independent generic points.

Trading T in for a smaller torus, and trading t in for a smaller value as well (but not too small) we can set this up so that we have:

- a finite group Σ operating on T, covering Sym_t,
- — and sitting inside a connected group H —
- such that T is the definable hull of a maximal 2-torus in H.

Let us simplify considerably.

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It seems reasonable that this action can be exploited to blow up T, and also T/T_∞.
There is a glaring hole in this argument.

Plugging a hole

The Setup

T inside G,
G connected,
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So we are done.

More results on torsion

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- Semisimplicity

  If G is connected, then every p-element is in a torus.
- Sylow theory

  For all primes p
- Weyl groups $N(T)/T$.

  If the Weyl group is nontrivial, it contains an involution.

  (Burdges-Deloro) If the group is minimal simple, the Weyl group is cyclic

Applications

1. Permutation Groups
2. Classification in odd type and low 2-rank
3. Bounds on 2-rank revisited?

Other aspects

1. The Borovik Program: Signalizer functor theory, strong embedding, black box group theory ...
2. Burdges unipotence theory and the Bender method
3. Generix strikes back [Nesin, Jaligot]
4. Conjugacy of Carter subgroups [Frécon]
5. Quasithin methods
    1. Amalgam method, representation theory (even type)
    2. Component analysis (odd type) [Borovik, Altseimer, Burdges]

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L^*-group theory in odd type (absolute bounds on 2-rank)

Control of actions of 2-tori on degenerate type groups.

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Bad groups and non-commutative geometry . . . ?