Structure/Nonstructure for Classes of Finite Models

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RUTGERS

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Dichotomies as algorithmic problems

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Thesis: these properties give *dichotomies* if the answer is computable.

1 WQO/¬WQO

2 Universality / Nonuniversality

WQO

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- No infinite descending sequences;
- No infinite antichains.

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The theorem of Robertson and Seymour . . . was proved using finitely many iterated applications of the "minimal bad sequence" method from well-quasi-ordering theory. It is shown [in FRS1987] that some such (impredicative) methods must be used . . .
[Friedman, Robertson, Seymour 1987]

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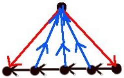
Tournaments: 

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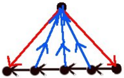
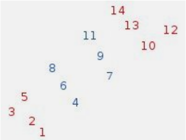
Tournaments:  Permutations: 

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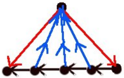
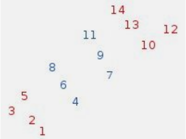
Tournaments:  Permutations: 

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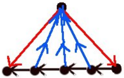
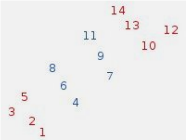
Tournaments:  Permutations: 

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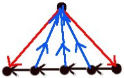
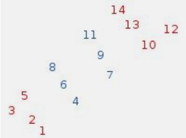
Tournaments:  Permutations: 

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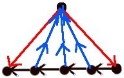
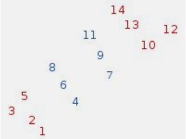
Tournaments:  Permutations: 

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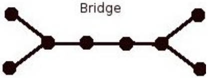
  I_1: Bridges: 

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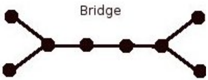
  I_1: Bridges: 

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Proposition

Among vertex-colored paths, the minimal antichains are quasi-periodic, that is they consist of a periodic part augmented by a first and last vertex which break the periodicity.

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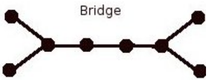
  I_1: Bridges: 

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Corollary

In the cases of graphs and colored paths, the isolated minimal antichains are dense, the associated ideals are effectively recognizable, and the recognition of wqo classes given by finitely many constraints is effective, in polynomial time.

A finiteness Theorem

Theorem (Cherlin-Latka 2000)

Let $\mathcal{Q}$ be a wellfounded quasiorder. Then for each k, there is a finite set Λ_k of minimal antichains, such that any non-wqo $\mathcal{Q}_{\mathcal{C}}$ with $|\mathcal{C}| \leq k$ allows one of the antichains in Λ_k (up to a finite set).

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Corollary

If the ideals $\mathcal{Q}^{<I}$ are computable for $I \in \Lambda_k$, then the decision problem for wqo with respect to $k+1$ constraints is decidable.

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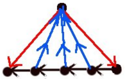

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- Permutation patterns - almost nothing is known.
- Better wqo theorems for substructures—Robertson/Seymour techniques?

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2. Universality / Nonuniversality

Around Universality

This time we consider countable $\mathcal{C}$-free structures and ask whether there is a universal one.

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Encoding by structures: a graph G with maximum vertex degree 2, tiling relations $T_i(u, v)$ on G^2, and a unary predicate A on G. There is a universal graph if and only if there is no tiling—then the components of G are finite, of bounded size.

A reduction theorem

Back to the case of forbidden substructures.

Theorem (Cherlin-Shi)

The universality problem for general relational systems in a finite language reduces to the case of graphs with a vertex coloring by two colors.

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Theorem (CSS)

If $acl_{\mathcal{C}}$ is locally finite, there is a univeral $\mathcal{C}$-free graph.

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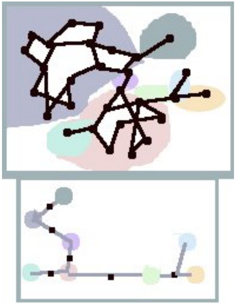
Blocks and trees:



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Conjecture

Let C be a graph obtained from a complete graph K by adjoining one path to each vertex. Then acl_C is locally finite and so there is a universal C-free graph.

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$S2S$?

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2. Ideas: Algebraic closure, automata theory, explicit combinatorics.