Around Homogeneous Universal Graphs

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Contents

Homogeneity

$A \simeq B \implies A \sim B$ under $\mathrm{Aut}(\Gamma)$

E.g. $(\mathbb{Q}, <)$

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Urysohn 1927 (Ph.D. 1921; d. 1924, aged 26): $\mathbb{U}$

Rado 1964: G

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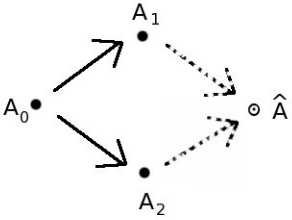

Amalgamation of Metric Spaces

1-point extensions: $A_i = A_0 \cup \{u_i\}$.

$d^+(u_1, u_2) = \min(d(u_1, a) + d(u_2, a))$
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$\mathbb{U}_0$: The universal homogeneous countable rational-valued metric space.
$\mathbb{U}$: The completion of $\mathbb{U}_0$.

Homogeneous Graphs and Digraphs

Henson 1971: G_n (K_n-free graph), its automorphisms and structure
Henson 1972: $D_{\neg\mathcal{T}}$ ($\mathcal{T}$-free digraph)

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Tools: Fraïssé, Finite Ramsey theorem

Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Color classes $c_1 \leq c_2 \leq c_1$*, densely colored; connections between pairs of color class components; triples.* Fraïssé for existence.

Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Kechris-Pestov-Todorcevic 2005:
Fraïssé+Ramsey+Top. Dynamics

Glasner: “This remarkable paper is a tour de force where three experts in disparate areas—model theory, structural Ramsey theory and topological dynamics—collaborate in creating a unified and beautiful theory.”

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Minimal flows: compact actions with every orbit dense.
Extremely amenable: no nontrivial minimal flow

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• The extremely amenable closed subgroups of Sym_∞ are exactly the groups of the form $Aut(\mathbb{A})$ with $\mathbb{A}$ the Fraïssé limit of a Fraïssé order class with the Ramsey property.
• If $\mathbb{A}$ is one of the following structures, then the universal minimal flow $M(G)$ of the group $G = Aut(\mathbb{A})$ is its action on the space of linear orderings of the universe of $\mathbb{A}_0$:

- G_n ($n \leq \infty$);
- $(\mathbb{N}, =)$;
- $\mathbb{U}_0$

Distance homogeneous graphs?

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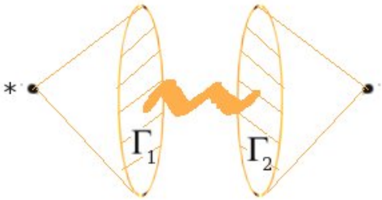

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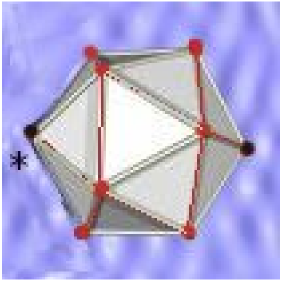

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   1. C_n $(n \leq \infty)$
   2. "Doubles"
   3. Tree-like (r-tree of s-cliques: $r, s \leq \infty$)
3. Fraïssé type
   - $\delta \leq d$;
   - Omit $(1, d)$-subspaces ($d \geq 3$);
   - Omit odd cycles up to order $2K + 1$;
   - Omit triangles of perimeter $\geq C$.

     Some interactions in these constraints.

Γ_1

Exceptional $\Gamma_1 \to$ Exceptional Γ.

Γ_1

Exceptional $\Gamma_1 \longrightarrow$ Exceptional Γ.

Difficulty: Γ_k

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Homogeneous metric space; not necessarily with the graph metric, because of the parity condition.

But (Γ_{k-1}, Γ_k) should be.
Extend the classification project?

Universal Graphs

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Data: Finitely many constraints $\mathcal{C}$ (finite, connected "forbidden" graphs).
Universal countable $\mathcal{C}$-free graph? ? Decidable ?

Universality and $\aleph_0$-categoricity

Existentially complete $\mathcal{C}$-free graphs.
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Existentially complete $\mathcal{C}$-free graphs.
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If the existentially complete countable graph is unique, then it is universal.
And there is an exact criterion for this in terms of the *algebraic closure*.

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• Forbid C_4. Then for points u, v at distance 2, the "midpoint" is a definable function $f(u, v)$. Such points are in the "definable closure" of u, v.

• Forbid a star S_k. Then for any u, the neighbors of u are "algebraic" over u: they lie in a u-definable finite set.

$\aleph_0$-categoricity and algebraic closure

Theorem (CSS 1999)

Let $\mathcal{C}$ be a finite set of forbidden graphs, T the theory of the existentially complete $\mathcal{C}$-free graphs. Then the following are equivalent.

1. *T has a unique countable model*
2. *The algebraic closure operator is locally finite.*

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Proof.

$\implies$: General nonsense (Ryll-Nardzewski, Engeler, Svenonius)

$\impliedby$: Close analysis: over any finite algebraically closed set, the set of types is finite. □

Applications: Cycles . . .

Conjectured by Menachem Kojman:

Theorem

If $\mathcal{C}$ is closed under homomorphism (i.e., the image of a constraint in $\mathcal{C}$ under graph homomorphism is $\mathcal{C}$-forbidden) then acl is degenerate and there is a universal $\mathcal{C}$-free graph.

Example. Odd cycles.

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Example. Odd cycles.

Theorem (Cherlin-Shi 1996)

For $\mathcal{C}$ a finite set of cycles the following are equivalent.

1. *There is a universal $\mathcal{C}$-free graph.*
2. *$\mathcal{C}$ consists of all odd cycles up to a fixed length.*

. . . and trees

Theorem (Cherlin-Shelah 2007)

For $\mathcal{C} = \{T\}$ a single tree, the following are equivalent.

1. *There is a universal $\mathcal{C}$-free graph.*
2. *The tree T is an extension of a path by at most one additional edge.*

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$\Longrightarrow$:

Shelah's idea: Pruning

To prune a tree T: T' is obtained by removing all leaves.

Pruning Trees

Lemma

If there is a T-free universal graph G then there is a T'-universal graph G^*, consisting of the vertices of G of infinite degree.

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Minimal trees: those which prune to a path or near-path.

(15 cases).

General Pruning

In general: Remove a minimal block-leaf. (Or a downward-closed family.)

Conjectures

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Conjecture

For a single connected constraint C, the problem of determining whether there is a universal C-free graph is algorithmically decidable.

Two Questions

- Is the generic triangle-free graph G_3 pseudofinite (i.e., are its properties shared by finite graphs)?

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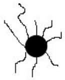
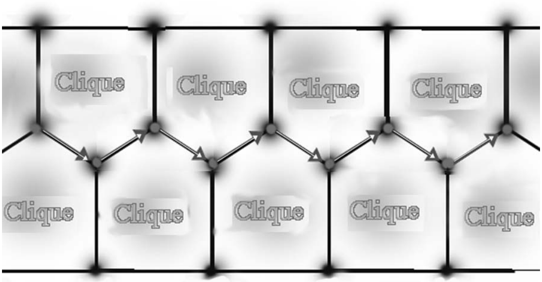
- The Hairy Ball Problem Let K be a finite graph consisting of a complete graph together with a single finite path attached to each vertex. Is there a universal K-free graph?



A Concrete Example

The Bouquet $K_5 \wedge K_5$



$(K_5 + K_5)$ - free

(Algebraic closure running along the mid-line)