Homogeneous Ordered Graphs

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Abstract

Theorem

All homogeneous ordered graphs are known.

Proof.

[Cherlin1998, Chap. IV] — as modified in http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf. □

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Definition (Urysohn, 1924, letter to Hausdorff)

Any isomorphism between finite parts is induced by an automorphism.

FRAÏSSÉ:

Homogeneous structures $\Gamma \iff$ Amalgamation Classes $\mathcal{A}$

$\mathcal{A} = \mathrm{Sub}(\Gamma)$; Γ is the *Fraïssé Limit* of $\mathcal{A}$

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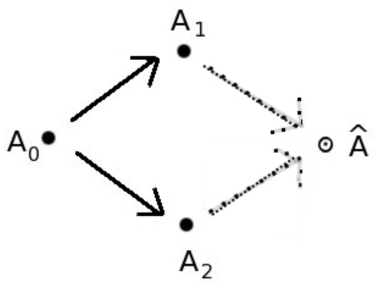

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If Γ is a homogeneous structure then the category $\mathrm{Sub}(\Gamma)$ of f.g. substructures has the amalgamation property and joint embedding.

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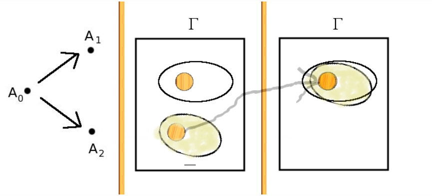

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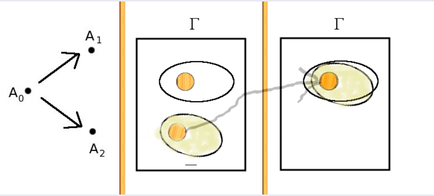

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- The Random Graph Γ_∞.
- The generic triangle-free graph Γ_3
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Structural Ramsey Theory

Theorem (Ramsey)

$$\mathrm{N} \to (\mathrm{B})^{\mathrm{A}}_{\mathrm{k}}$$

Given $\mathrm{A}, \mathrm{B}, \mathrm{k}$ *find* N*:*

Coloring $\binom{[1,\ldots,\mathrm{N}]}{\mathrm{A}}$ *makes some* B*-set* A*-monochromatic*

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Examples

Finite graphs, finite directed graphs, finite triangle-free graphs

NO

Finite orders, finite ordered graphs, finite ordered triangle-free graphs, finite metric spaces

YES

Compact Flows

$\mathcal{L} \iff \mathbb{Q} \iff \mathrm{Sym}(\infty)$ (with topology)

PESTOV

Ramsey's Theorem for $\mathcal{L}$	$\leftrightarrow$	Fixed point property for compact $\mathrm{Aut}(\mathbb{Q})$-flows

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Remark

If $\mathrm{Aut}(\Gamma)$ is has fixed points on compact flows then Γ has a definable linear order.

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Bodirsky has drawn attention to the following question—his motivation coming from applications to computer science.

Question

Given a homogeneous structure in a finite relational language, is there a homogeneous expansion with the same properties, and with a structural Ramsey theorem?

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Lessons of KPT

- Take ordered structures seriously.
- Take metric spaces seriously.

From my perspective this raises two problems in particular.

- Classify the homogeneous ordered graphs (Nguyen Van Thé, 2012; avoided by Macpherson [2010] and Cherlin [2011])
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Remark on [Cherlin1998, Appendix]:

We described 27 homogeneous structures with 4 nontrivial symmetric 2-types, not accounted for by general principles. 18 can be interpreted as metrically homogeneous, three are generic liftings of a metrically homogeneous graph of diameter 3 by generically splitting a type, and 6 remain unexplained.

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The present talk deals only with the first problem.

Some Classifications

All the homogeneous structures of the following types (and others) have been classified.

- Homogeneous Permutations: CAMERON [2003]
- Partial Orders SCHMERL [1979]
- Tournaments LACHLAN [1984]
- Graphs LACHLAN/WOODROW [1980]
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But not all

The Generic Local Order

An interesting exceptional example is the *generic local order*.

Definition

A *local order* is a tournament such that the in-neighbors and the out-neighbors of any vertex form a linear order.

Theorem (Lachlan)

The infinite homogenous tournaments are

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Summing up

- Classification theorems are one way to find nontrivial examples of the theory (e.g., in the case of finite simple groups)
- We need more examples to test a number of very broad conjectures.
  In the context of structural Ramsey theory, we prefer ordered structures.
  We can get some by generically ordering unordered structures, but this may miss the subtle cases.
- There are good classification methods which have been applied in the unordered cases.

Hence Lionel Nguyen Van Thé's question: the unordered symmetric graph case is a paradigm, can we add ordering?

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Hence Lionel Nguyen Van Thé's question: the unordered symmetric graph case is a paradigm, can we add ordering?

Can we add ordering?

This question raises the following issues.

Me: Why haven't we already done this?

Altınel: Haven't you already done this?

These are two good questions.
We will address them in order.

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Why is this Hard?

A few facts.

- There are five homogenous tournaments, and only three of them are infinite. Lachlan's proof of this introduced two fundamental techniques to the subject.
- There are infinitely many homogenous graphs, which may be listed explicitly. The Lachlan/Woodrow proof has not been adapted to asymmetric situations to date.
- There are uncountably many homogeneous digraphs, which may also be listed explicitly, using a real parameter. This used the techniques introduced for the case of tournaments—only needed there to characterize **one** structure.

This suggests we might want to warm up on the problem of homogeneous ordered tournaments . . .

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If we had embarked on the homogenous ordered tournament problem knowing only the homogenous tournaments we would have had to come up with the Lachlan/Woodrow classification somehow along the way.

Alternatively, if we had embarked on the homogeneous ordered graph problem without considering the homogeneous tournaments we would have had to reproduce Lachlan's work, which was one the fundamental advances in methodology!

Conclusion: A homogeneous ordered graph need not be an ordered homogeneous graph.

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Why is this Easy?

Returning to Altınel's question: Haven't you done this already?

This turns out to be the key question. Here "you" needs to be construed broadly, as including Lachlan and Woodrow.

Haven't you done this already?

There are two proofs of the classification of the homogeneous graphs.

Lachlan/Woodrow 1980 Introduced subtle inductive methods relating to amalgamation classes.

Cherlin 1998, Chap. 4 A proof based on Lachlan's ideas from 1984 involving use of Ramsey's Theorem.

In the second proof the sequence of ideas was: Generalize from tournaments to directed graphs, then specialize back to the symmetric case.

Cherlin 1998, Chap. 4

> *"The proof given here is more complex than the one given [by Lachlan/Woodrow], but it generalizes . . ."*

I did not realize at the time that that sentence could have ended with the words "to the ordered case."

And I have argued above that this is unlikely.

Objection The proof given in 1998 can only show at best that homogeneous ordered graphs are generic linear extensions of homogeneous graphs (allowing for sporadic exceptions, such as the imprimitive cases).

But this is **false**.

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Objection Overruled

What if we treat the homogeneous ordered tournaments as sporadic?
What does this mean?

We must account for ordered expansions of $\mathbb{Q}$ and $\mathbb{S}$ (the generic local order).

- Cameron treated linear expansions of $\mathbb{Q}$ (homogeneous permutations.
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Graphs vs. Tournaments

Corollary

The classification of homogeneous tournaments with trivial acl *follows from the classification of homogeneous ordered graphs.*

Proof.

Generically order the tournament and view it as a homogenous ordered graph. □

The three cases

We view our structures as ordered tournaments or as ordered graphs, interchangeably.
Replacing the structure by its complement, we may suppose that the graph contains an infinite independent set I_∞ (as an ordered tournament, this is $\mathbb{Q}$ with its usual ordering, twice).

Special	Omits some ordered form of the 3-cycle C_3.
Sporadic	Realizes both ordered forms of C_3 ($\vec{P}_3$, $\vec{P}_3^c$) and $\vec{I}_\infty$, but omits $\vec{I}_1 \perp \vec{P}_3$.
Generic	Realizes $\vec{I}_1 \perp \vec{P}_3$, $\vec{P}_3^c$, $\vec{I}_\infty$.

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For more information

http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf.

Special Cases

Theorem

Let Γ be a homogeneous ordered tournament which omits some ordered form of the 3-cycle C_3. Then:

- *If Γ omits both forms of C_3, it is a homogeneous permutation (by definition)*
- *If Γ omits exactly one ordered form of C_3 then Γ or its complement is a linear extension of a homogeneous partial order with strong amalgamation, namely one of the following:*
  - *If Γ is primitive, the linear order may be the generic linear order with convex equivalence classes;*
  - *Γ may be the generic linear extension of a homogeneous partial order with strong amalgamation.*

The Sporadic Case

Theorem

Let Γ be a homogeneous ordered tournament which contains both ordered forms of C_3, as well as $\vec{I}_\infty$, but not $[\vec{I}_1 \to \vec{C}_3^+]$. Then Γ is the generically ordered generic local order.

Proof.

Brute force. $\square$

The Generic Case

Lemma

Let Γ be a homogeneous ordered graph which contains $\vec{P}_3^c$, $\vec{I}_1 \perp \vec{P}_3$, and $\vec{I}_\infty$. Then the underlying graph of Γ is either a Henson graph or the Rado graph, and Γ is generically ordered.

This is where Lachlan's methods come into play. A sketch of the argument is found in [Cherlin1998, Chap. IV], with the following proviso: "1-types" xA should be limited to "initial 1-types," meaning those for which $x < A$. Similarly, the direct sum construction $A + B$ should be interpreted as $A \perp B$, with $A < B$.

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The Ramsey Argument

$\mathcal{A}(n)$: $\{\vec{I}_1 \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_n\} \cup \{\vec{I}_k \mid k < \infty\}$

For any nontrivial 2-type r, $\mathcal{A}^r$ is the set of all finite ordered graphs A such that

For $R < A$ with R r-Ramsey and RA $\mathcal{A}$-constrained we have

$$RA \in \mathcal{A}$$

For any r, $\mathcal{A}^r$ is an amalgamation class.

Theorem

If $\mathcal{A}(n) \subseteq \mathcal{A}$ then for some r, $\mathcal{A}(n) \subseteq \mathcal{A}^r$.

Proof.

[1998, Chap. IV, updated] □

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Proof.

[1998, Chap. IV, updated] $\square$

Lachlan's Punchline

The Main Theorem follows by a trivial induction:

$$\text{Given } A,\ K_{n+1}\text{-free}$$
$$R := \{\min A\},\ A' := A \setminus R$$
$$R < A'$$
$$R \text{ is } r\text{-Ramsey (any } r)$$
$$[\text{Ind}]\ A' \in \mathcal{A}^r \implies A = RA' \in \mathcal{A}$$