Homogeneous Ordered Graphs

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

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Abstract

Theorem

All homogeneous ordered graphs are known.

Proof.

[Cherlin1998, Chap. IV] — as modified in http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf. □

1. Abstract

2. The Problem in Context
   - Homogeneity
   - Structural Ramsey Theory and Topological Dynamics
   - A Question
   - Classification Theorems
   - Examples

3. Homogeneous Ordered Graphs
   - Why Haven't We Done This Already?
   - A Sketch of the Proof

Homogeneity

Definition (Urysohn, 1924, letter to Hausdorff)

Any isomorphism between finite parts is induced by an automorphism.

FRAÏSSÉ:

Homogeneous structures $\Gamma \iff$ Amalgamation Classes $\mathcal{A}$

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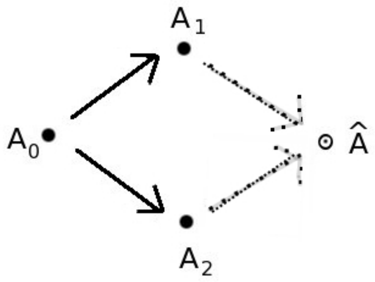

Examples

- The rational order $\mathbb{Q}$.
- The Random Graph Γ_∞.
- The generic triangle-free graph Γ_3
- The generically ordered version of any of the above (e.g. for 1: the generic permutation)

Structural Ramsey Theory

Theorem Template (Structural Ramsey)

$$\mathcal{N} \to (\mathcal{B})^{\mathcal{A}}_{k}$$

Given $\mathcal{B}, \mathcal{A}, k$ find $\mathcal{N}$: Coloring $\binom{\mathcal{N}}{\mathcal{A}}$ makes some $\mathcal{B}$ be $\mathcal{A}$-monochromatic.

Finite graphs, finite directed graphs, finite triangle-free graphs

NO

Finite orders, finite ordered graphs, finite ordered triangle-free graphs, finite metric spaces

YES

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Outline

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Structural Ramsey Theory and Topological Dynamics

A Question

Classification Theorems

Examples

Homogeneous Ordered Graphs

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Compact Flows

$\mathcal{A} \iff \Gamma \iff \mathrm{Aut}(\Gamma)$ (with topology)

KECHRIS/PESTOV/TODORČEVIĆ:

Structural Ramsey Theory for $\mathcal{A}$ with order $\leftrightarrow$ Fixed point property for compact $\mathrm{Aut}(\Gamma)$-flows

Remark

If $\mathrm{Aut}(\Gamma)$ is has fixed points on compact flows then Γ has a definable linear order.

Because $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{L}(\Gamma) \subseteq 2^{\Gamma\times\Gamma}$

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From Homogeneity to Ramsey Theory?

Bodirsky has drawn attention to the following question—his motivation coming from applications to computer science.

Question

Given a homogeneous structure in a finite relational language, is there a homogeneous expansion with the same properties, and with a structural Ramsey theorem?

What are some good test cases?

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Lessons of KPT

- Take ordered structures seriously.
- Take metric spaces seriously.

From my perspective this raises two problems in particular.

- Classify the homogeneous ordered graphs (Nguyen Van Thé, 2012; avoided by Macpherson [2010] and Cherlin [2011])
- Classify the metrically homogeneous graphs (Cameron, 1998)

The present talk deals only with the first problem.

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Remark on [Cherlin1998, Appendix]:

We described 27 homogeneous structures with 4 nontrivial symmetric 2-types, not accounted for by general principles. 18 can be interpreted as metrically homogeneous, three are generic liftings of a metrically homogeneous graph of diameter 3 by generically splitting a type, and 6 remain unexplained.

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Some Classifications

All the homogeneous structures of the following types (and others) have been classified.

- Homogeneous Permutations: CAMERON [2003]
- Partial Orders SCHMERL [1979]
- Tournaments LACHLAN [1984]
- Graphs LACHLAN/WOODROW [1980]
- Directed Graphs CHERLIN [1998]
- Vertex colored partial orders (Torrezao de Sousa/Truss) [2008]
- Metrically homogeneous graphs with triangle constraints (Cherlin) [20??]

Most examples are natural, e.g. the Henson graphs (generic K_n-free).

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But not all

The Generic Local Order

An interesting exceptional example is the *generic local order* $\mathbb{S}$.

Definition

A *local order* is a tournament such that the in-neighbors and the out-neighbors of any vertex form a linear order.

Theorem (Lachlan)

The infinite homogenous tournaments are

(a) *The rational order*

(b) *The generic local order*

(c) *The generic tournament*

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Summing up

- Classification theorems are one way to find nontrivial examples of the theory (e.g., in the case of finite simple groups)
- We need more examples to test a number of very broad conjectures.
  In the context of structural Ramsey theory, we prefer ordered structures.
  We can get some by generically ordering unordered structures, but this may miss the subtle cases.
- There are good classification methods which have been applied in the unordered cases.

Hence Lionel Nguyen Van Thé's question: the unordered symmetric graph case is a paradigm, can we add ordering?

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Hence Lionel Nguyen Van Thé's question: the unordered symmetric graph case is a paradigm, can we add ordering?

Can we add ordering?

This question raises the following issues.

Me: Why haven't we already done this?

Altınel: Haven't you already done this?

These are two good questions.

We will address them in order.

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Why is this Problem Hard?

A few facts.

- There are five homogenous tournaments, and only three of them are infinite. The difficulty is to characterize the generic tournament (Lachlan's Ramsey argument).
- There are infinitely many homogenous graphs, which may be listed explicitly. The Lachlan/Woodrow proof has not been adapted to asymmetric situations.
- There are uncountably many homogeneous digraphs, which may also be listed explicitly, using a real parameter. This Lachlan's Ramsey method—originally used to characterize **one** structure.

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Warming Up

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Proof.

$$R \leftrightarrow S \triangle <$$

□

Warming Up

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Remark

The classes of homogeneous ordered tournaments and homogeneous ordered graphs are the same.

This is disconcerting for the classifier.

Stupeur et Tremblements

If we had embarked on the homogenous ordered tournament problem knowing only the homogenous tournaments we would have had to come up with the Lachlan/Woodrow classification somehow along the way.

Alternatively, if we had embarked on the homogeneous ordered graph problem without considering the homogeneous tournaments we would have had to reproduce Lachlan's work, which was one the fundamental advances in methodology!

Conclusion: A homogeneous ordered graph need not be an ordered homogeneous graph.

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Why is this Problem Easy?

Returning to Altınel's question: Haven't you done this already?

This turns out to be the key question. Here "you" needs to be construed broadly, as including Lachlan and Woodrow.

Haven't you done this already?

There are two proofs of the classification of the homogeneous graphs.

Lachlan/Woodrow 1980 Introduced subtle inductive methods relating to amalgamation classes.

Cherlin 1998, Chap. 4 A proof based on Lachlan's ideas from 1984 involving use of Ramsey's Theorem.

In the second proof the sequence of ideas was: Generalize from tournaments to directed graphs, then specialize back to the symmetric case.

Cherlin 1998, Chap. 4

As I said then,

> *"The proof given here is more complex than the one given [by Lachlan/Woodrow], but it generalizes . . . "*

But I did not realize at the time that that sentence could have ended with the words "to the ordered case."

And I have argued above that this is unlikely.

Objection The proof given in 1998 can only show at best that homogeneous ordered graphs are generic linear extensions of homogeneous graphs (allowing for sporadic exceptions, such as the imprimitive cases).

But this is **false**.

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Objection Overruled

What if we treat the homogeneous ordered tournaments as sporadic?

What does this mean?

We must account for ordered expansions of $\mathbb{Q}$ and $\mathbb{S}$ (the generic local order).

- Cameron treated linear expansions of $\mathbb{Q}$ (homogeneous permutations.
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- This leaves $\mathbb{S}$ to be captured along the way.

This works.

Graphs vs. Tournaments

Corollary

The classification of homogeneous tournaments with trivial acl *follows from the classification of homogeneous ordered graphs.*

Proof.

Generically order the tournament and view it as a homogenous ordered graph. □

The three cases

We view our structures as ordered tournaments or as ordered graphs, interchangeably.
Replacing the structure by its complement, we may suppose that the graph contains an infinite independent set I_∞ (as an ordered tournament, this is $\mathbb{Q}$ with its usual ordering, twice).

Special Omits some ordered form of the 3-cycle C_3.

Sporadic Realizes both ordered forms of C_3 ($\vec{P}_3$, $\vec{P}_3^c$) and $\vec{I}_\infty$, but omits $\vec{I}_1 \perp \vec{P}_3$.

Generic Realizes $\vec{I}_1 \perp \vec{P}_3$, $\vec{P}_3^c$, $\vec{I}_\infty$.

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Generic	Realizes $\vec{\mathrm{I}}_1 \perp \vec{\mathrm{P}}_3$, $\vec{\mathrm{P}}_3^c$, $\vec{\mathrm{I}}_\infty$.

Special Cases

Theorem

Let Γ be a homogeneous ordered tournament which omits some ordered form of the 3-cycle C_3. Then:

- *If Γ omits both forms of C_3, it is a homogeneous permutation (by definition)*
- *If Γ omits exactly one ordered form of C_3 then Γ or its complement is a linear extension of a homogeneous partial order with strong amalgamation, namely one of the following:*
  - *If Γ is primitive, the linear order may be the generic linear order with convex equivalence classes;*
  - *Γ may be the generic linear extension of a homogeneous partial order with strong amalgamation.*

The Sporadic Case

Theorem

Let Γ be a homogeneous ordered tournament which contains both ordered forms of C_3, as well as $\vec{I}_\infty$, but not $[\vec{I}_1 \to \vec{C}_3^+]$. Then Γ is the generically ordered generic local order.

Proof.

Brute force. □

The Generic Case

Theorem (Generic Case)

Let Γ be a homogeneous ordered graph which contains $\vec{P}_3^c$, $\vec{I}_1 \perp \vec{P}_3$, and $\vec{I}_\infty$. Then the underlying graph of Γ is either a Henson graph or the Rado graph, and Γ is generically ordered.

This is where Lachlan's methods come into play.

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Proof Template

We follow [Cherlin1998, Chap. IV], with the following changes:

- The disjoint union of graphs A, B becomes their ordered disjoint union with $A < B$.
- "1-types" x/A are assumed to have $x < A$ or occasionally $A < x$. We call these **initial** or **terminal** 1-types.

The Main Theorem Rephrased

Theorem (Main Theorem)

$$\mathcal{A}(n) \implies \mathcal{K}_n$$

Notation

- *$\mathcal{K}_n$ is the set of $\vec{K}_{n+1}$-free finite ordered graphs.*
- *$\mathcal{A}(n)$ is a set of generators for $\mathcal{K}_n$, specifically*

$$\{\vec{I} \perp \vec{P}_3, \vec{P}_3^c, \vec{K}_n\} \cup \{\vec{I}_k \mid k < \infty\}$$

- *$\mathcal{A} \implies \mathcal{B}$ ($\mathcal{A}$ generates $\mathcal{B}$ means:*

  Whenever an amalgamation class of finite ordered graphs contains $\mathcal{A}$, then it contains $\mathcal{B}$.

Exegesis

The theorem says that a homogeneous ordered graph which embeds the ordered graphs in $\mathcal{A}(n)$ but not $\mathcal{A}(n+1)$ is the generically ordered Henson graph, while a homogenous ordered graph that embeds the ordered graphs in all $\mathcal{A}(n)$ is the generically ordered random graph.

Proof Outline

Proposition

Fix n and suppose that $\mathcal{A}$ is an amalgamation class of finite ordered graphs containing $\mathcal{A}(n)$. Then the following four statements hold.

1. *Any ordered direct sum $A \in \perp \mathcal{A}(n)$ belongs to $\mathcal{A}$.*
2. *Any extension vA with $v < A$ of $A \in \perp \mathcal{A}(n)$ belongs to $\mathcal{A}$.*
3. *For some 2-type r any r-linear extension LA of $A \in \mathcal{A}(n)$ belongs to $\mathcal{A}$.*
4. *$\mathcal{K}_n \subseteq \mathcal{A}$.*

Notation

$\perp \mathcal{A}(n)$ consists of all $\perp_1^k A_i$ with $A_i \in \mathcal{A}(n)$

Proof Outline

Proposition

1. *Any ordered direct sum $A \in \perp \mathcal{A}(n)$ belongs to $\mathcal{A}$.*

Proof.

Induction on the number of factors. If $A \subseteq \Gamma$ then $A^{\perp} = \{v > A \mid A \perp v\}$ is homogeneous. $\square$

Proof Outline

Lemma

$$2 \implies 3 \implies 4$$

2. *Any extension vA with* $v < A$ *of* $A \in \perp \mathcal{A}(n)$ *belongs to* $\mathcal{A}$.
3. *For some 2-type r any r-linear extension LA of* $A \in \mathcal{A}(n)$ *belongs to* $\mathcal{A}$.
4. $\mathcal{K}_n \subseteq \mathcal{A}$.

Proof.

$(2 \implies 3)$: Ramsey's Theorem

$(3 \implies 4)$: Trivial, as it turns out . . . . □

Lachlan's Punchline

The Main Theorem follows by a trivial induction:

Lemma

$$3 \implies 4$$

3. *For some 2-type r any r-linear extension LA of $A \in \mathcal{A}(n)$ belongs to $\mathcal{A}$.*
4. $\mathcal{K}_n \subseteq \mathcal{A}$.

Proof.

$\mathcal{A}(n) \subseteq \mathcal{A}$
$\mathcal{A}' = \{X \mid \text{All } LX \text{ are in } \mathcal{A}\}$
$\mathcal{A}(n) \subseteq \mathcal{A}'$

$A \in \mathcal{K}_n$
$v = \{\min A\}$, $L = \{v\}$,
$A' = A \setminus R$
$A = LA'$, $L < A'$

$A' \in \mathcal{A}'$, so $A = LA' \in \mathcal{A}$

□

One more thing

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1. *Any ordered direct sum $A \in \perp \mathcal{A}(n)$ belongs to $\mathcal{A}$.*
2. *Any extension vA with $v < A$ of $A \in \perp \mathcal{A}(n)$ belongs to $\mathcal{A}$.*

How to prove (2)? Like (1)—but that's another story, and leads to complications.

Details at

http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf.