Classification of Homogeneous Combinatorial Structures

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1. Introduction
2. Amalgamation
3. Classification
4. Open Problems

Homogeneity

Definition

A structure is homogeneous iff every isomorphism between f.g. substructures is induced by an automorphism.

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Examples

- $(\mathbb{Q}, <)$
- A regular tree, as a metric space.
- The random graph

Regular trees as metric spaces

Recovering the convex closure from the metric.

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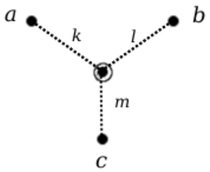

$$P = |(a, b, c)| = 2(k + l + m);\ m = P/2 - d(a, b)$$

The Random Graph

Alice's Restaurant Axioms

$\forall x_1, \ldots, x_n$ You can get anything you want

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Hence: *Uniqueness and Homogeneity*

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Corollary (0-1 law; Fagin76, GKLT69)

Any first order property of graphs has asymptotic probability 0 or 1 in large random graphs.

Classifications

- Finite homogeneous for a finite relational language (Lachlan): finitely many families, each consisting of approximations to an infinite limit;
- Some binary relational structures (ad hoc)

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Method	Example	Reference
Structural Analysis	Colored P. O.	T. de Sousa/Truss 2008
"	Permutation patterns	Cameron 2002
Artful Induction	Graphs	Lachlan/Woodrow 1980
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Some Open Cases

- metrically homogeneous graphs (Cameron, 1998)
- k-dimensional permutation patterns (Cameron, 2002)

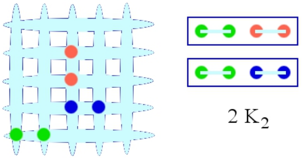
"Sporadic" finite structures

Theorem (Sheehan 74, Gardiner 76)

The finite homogeneous graphs are:

- $m \cdot K_n$ *and its complement;*
- *The pentagon* C_5*;*
- *The "grid"* $K_3 \otimes K_3 = L[K_{3,3}]$

$K_n \otimes K_n$



Grids and Cycles

Varying the language (Lachlan's theory).

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On the other hand, the n-cycles C_n remain sporadic forever. They are metrically homogeneous but the number of binary relations involved is unbounded.

The finite primitive case

Question

Can one classify the finite primitive structures homogeneous for a language of bounded arity?

The binary case: (known examples)

- Equality;
- C_n, or $\vec{C}_n$;
- $[\mathbb{F}_{q^2} \cdot \ker(N)] \cdot \langle \mathrm{Fr}_q \rangle$

(O'Nan-Scott-Aschbacher?—cf. Saracino 1996–7)

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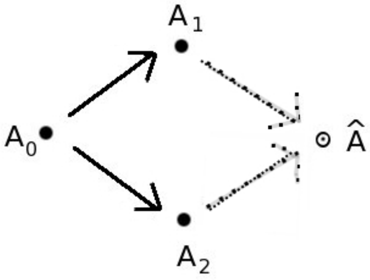
The amalgamation property



The amalgamation property

Remark (Fraïssé)

If Γ *is a homogeneous structure then the category* $\mathrm{Sub}(\Gamma)$ *of f.g. substructures has the amalgamation property and joint embedding.*

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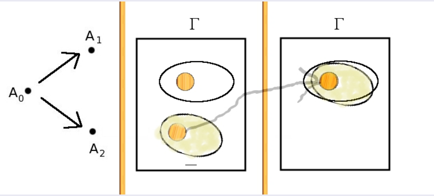

□

There is a converse . . .

The Fraïssé limit

Definition (Amalgamation Class)

A set $\mathcal{A}$ of f.g. structures is an amalgamation class if

- It is closed under isomorphism and substructure;
- It has the joint embedding and amalgamation properties

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Example

$(\mathbb{Q}, <)$ is the Fraïssé limit of the class $\mathcal{L}$ of finite linear orders.

Examples

- The generic partial order $\mathcal{P}$
- The generic K_n-free graph Γ_n [Henson 71]
- The generic $\mathcal{T}$-free directed graph [Henson 72]
- The rational Urysohn space $\mathbb{U}_0$ [Urysohn 1924]

Examples

- The generic partial order $\mathcal{P}$
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Fréchet's problem: is there a universal separable complete metric space?
Urysohn: Let $\mathbb{U}$ be the completion of the rational Urysohn space $\mathbb{U}_0$.

> *...in addition [it] satisfies a quite powerful condition of homogeneity: the latter being, that it is possible to map the whole space onto itself (isometrically) so as to carry an arbitrary finite set* M *into an equally arbitrary set* M_1*, congruent to the set* M*.*

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Colored partial orders

Theorem (Schmerl 1979)

A nontrivial homogeneous partial order is either a composition $\mathrm{I_n}[\mathbb{Q}]$ or $\mathbb{Q}[\mathrm{I_n}]$, or the generic partial order $\mathcal{P}$.

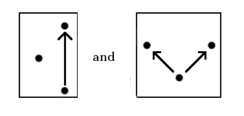
Theorem (Torrezão de Sousa, Truss 2008)

A homogeneous countably vertex colored partial order is built from generically colored components by assembly along a skeleton, which is a countable partial order with labels on edges indicating the isomorphism type of each pair of components.

Amalgamation arguments

Lemma

If a homogeneous partial order contains



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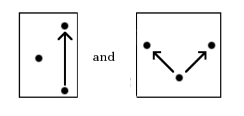

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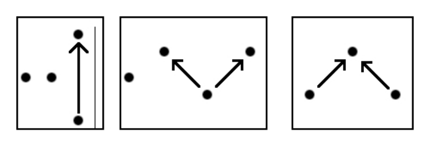
Proof.

Step 1. $+I_1$ and duality.



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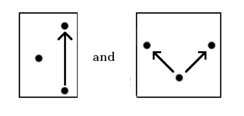

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Proof.

Step 2. Fan-in and fan-out.

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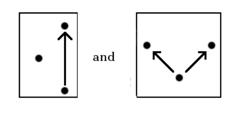

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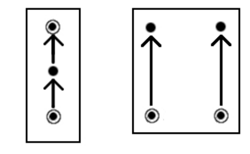
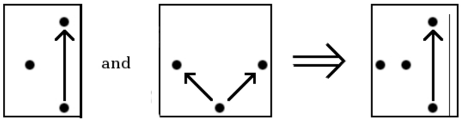
Step 3. Forced amalgamations



□

Explicit Amalgamation

Claim (Step 1)



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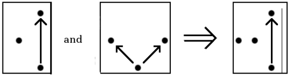

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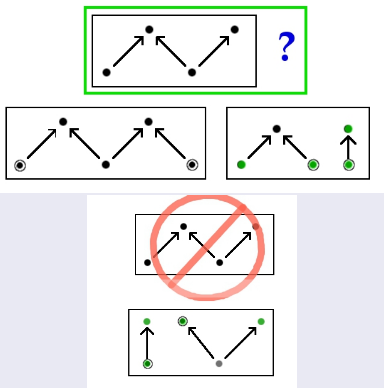


P.O. with vertex colors

Colors $\mathbb{C}$.

P.O. with vertex colors

Colors C.

Definition

$c_1 \leq c_2$ if $\exists x_1 \leq x_2$ of those colors.

Remark

Transitive!

$c \sim c'$: $c \leq c' \leq c$; $C/\sim$ is a partially ordered set.

P.O. with vertex colors

Colors C.

Definition

$c_1 \leq c_2$ if $\exists x_1 \leq x_2$ of those colors.

Remark

Transitive!

The components of Γ are the vertices whose colors belong to a fixed color class.

Lemma (1 Component)

The components are generically colored homogeneous partial orders.

Homogeneous Permutation Patterns

Definition

A permutation is a structure consisting of two linear orders.

The isomorphism types are the permutation patterns.

Theorem (Cameron 2002)

The nontrivial primitive homogeneous permutations are

- I $(<_2 = <_1)$ *and* I^{op} $(<_2 = <_1^{op})$*; or*
- *Generic.*

The imprimitive homogeneous permutations are compositions of primitive ones: $\mathrm{I}[\mathrm{I}^{op}]$, $\mathrm{I}^{op}[\mathrm{I}]$

(Main Lemma)

If a homogeneous permutation contains all permutation patterns of order 3, then it contains all patterns.

Homogeneous graphs

Theorem (Lachlan/Woodrow 1980)

The homogeneous graphs are as follows:

- C_5 *and* $K_3 \otimes K_3$
- $I_m[K_n]$ *and* $K_n[I_m]$ *(compositions)*
- *The generic* K_n*-free graph* Γ_n*, or its complement;*
- *The random graph* Γ_∞

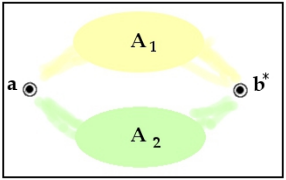
Reduction: w.l.o.g. Γ contains $I_\infty, I_1 \oplus K_2, P_2$.
Target: Some Γ_n $(n \leq \infty)$.

(Alice's Restaurant Lemma)

If the "generators" $I_\infty, I_1 + K_2, P_2$ *occur as well as* K_n*, then any finite graph omitting* K_{n+1} *occurs.*

Induction fails

$|A| = k$.
$a \in A$, (a, b) an edge, (a, b') a nonedge. $A_1 = A \setminus \{a, b\}$, $A_2 = A \setminus \{a, b'\}$.
$B = A_1 \oplus A_2$. Amalgamating $B \cup \{a\}$ with $B \cup \{b^*\}$ will force A.



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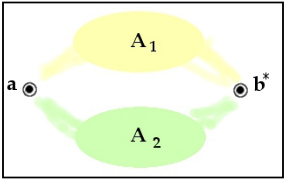
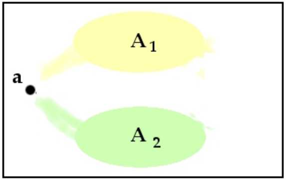


Factors:



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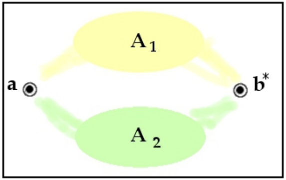

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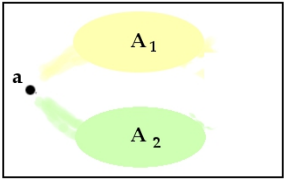

How can we make this work?

Greedy Induction

(Main Lemma′)

For any finite A *omitting* $\mathrm{K_{n+1}}$

> *If* H *is a consequence of the generators and* $\mathrm{a} \in \mathrm{A}$, $\mathrm{a}' \in \mathrm{H}$ *then the almost disjoint sum* $\mathrm{A} \oplus_{\mathrm{a}=\mathrm{a}'} \mathrm{H}$ *is a consequence of the generators.*

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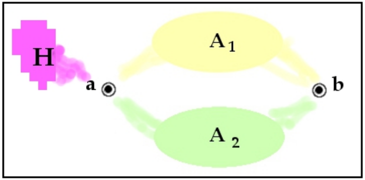
Now the amalgamation looks like this:



Greedy Induction

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Amalgamation

Classification

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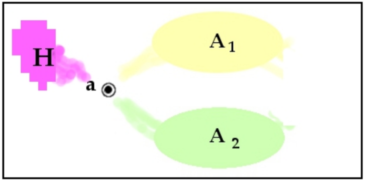
Main Factor:



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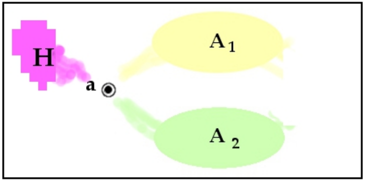

$H \implies (A_2 \oplus_a H) \implies A_1 \oplus_a (A_2 \oplus_a H)$ (by induction)

2nd factor: disjoint union. Explicit amalgamation arguments

Explicit Amalgamation Arguments

Question (Main Classification Problem—Lachlan)

Given finitely many positive constraints $A_1, \ldots, A_k$ *and negative constraints* $B_1, \ldots, B_\ell$, *is there a homogeneous structure meeting the constraints?*

Is this decidable?

Tournaments: The Ramsey Method

A tournament is a local order if for each vertex v the left and right sides v^- and v^+ are linear orders (transitive).
The homogeneous local orders are L_1, $\vec{C}_3$, $\mathbb{Q}$, and the generic local order $\mathbb{Q}^*$.

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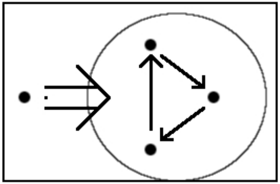
Theorem (Lachlan 1984)

The homogeneous tournaments are the homogeneous local orders and the generic tournament.

The "generator" $[L_1, \vec{C}_3]$



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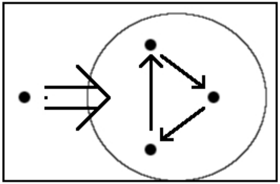

$[L_1, C_3]$

(Main Lemma)

$[L_1, \vec{C}_3] \implies$ *Everything*

Tournaments: The Ramsey Method

(Main Lemma)

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Step 1. Duality:
$[\mathrm{L}_1, \vec{\mathrm{C}}_3] \implies [\vec{\mathrm{C}}_3, \mathrm{L}_1]$
Tournaments omitting $[\vec{\mathrm{C}}_3, \mathrm{L}_1]$ have the form $[\mathrm{L}, \mathrm{S}]$ with L linear and S a local order. In the homogeneous case, T must be one or the other.

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Step 2. Linear extensions

$$\mathcal{A}^* = \{A : \text{All } A \cup L \text{ lie in } \mathcal{A}\}$$

Lemma

$\mathcal{A}^*$ is an amalgamation class.

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Therefore it suffices to prove: $[L_1, \vec{C}_3] \in \mathcal{A}^*$.

The Ramsey Argument

Stacks: $\mathrm{L}[\mathrm{A}]$ is a stack of A's.

Lemma

Assume every 1-point extension of a stack of A's is in $\mathcal{A}$. Then A is in $\mathcal{A}^*$.

Proof.

Amalgamate many 1-point extensions.

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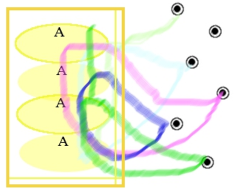

Amalgamating over a stack

A copy of L will appear. □

Stacks

Lemma

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$A = \vec{C}_3$.

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□

Final version: if $\mathbb{T} = (T_1, T_2)$ is an ample 2-tournament, and $A \subseteq T_1$, $A \simeq \vec{C}_3$, then

$$(A'(T_1), A^p(T_2)) \text{ is an ample 2-tournament.}$$

[Finitized]

The Case of Directed Graphs

Theorem

If Γ is a primitive homogeneous directed graph then Γ is one of the following.

- *A tournament or independent set of vertices;*
- *A local partial order $\mathbb{S}(2)$, $\mathcal{P}$, or $\mathcal{P}(3)$.*
- *Γ_{I_n} or $\Gamma_{\mathcal{T}}$ (Henson digraphs).*

Proof.

As for tournaments, allowing for some ambiguity in the Ramsey argument.
$\mathcal{A}^r = \{A : \text{Every } r\text{-Ramsey extension of } A \text{ lies in } \mathcal{A}\}$.
Instead of 1-point extensions of stacks of generators, we use disjoint sums of generators. □

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Some Classification Problems

- Homogeneous k-dim. permutations $(<_1, \ldots, <_k)$. (Compositions of generic for $\leq k$ linear orders?)
- Finite primitive binary homogeneous structures (O'Nan-Scott-Aschbacher)
- Metrically Homogeneous Graphs.

Metrically Homogeneous Graphs

The known metrically homogeneous graphs are of the following forms.

- Homogenous Graphs (Lachlan/Woodrow)
- The n-gon C_n, or an antipodal double of C_5 or $K_3 \otimes K_3$.
- Tree-like graphs $T_{r,s}$: r-fold branching of s-cliques.
- $\Gamma^{\delta}_{K,C,\mathcal{S}}$ where
  - δ is the diameter
  - $K = (K_1, K_2)$ controls triangles of odd perimeter
  - $C = (C_0, C_1)$ controls triangles of large perimeter ($\geq 2\delta$)
  - $\mathcal{S}$ is a Henson-style constraint involving $(1, \delta)$-subspaces.
- An antipodal variation of the previous example, $\Gamma^{\delta}_{a,n}$ omitting K_n and some related subgraphs.

The evidence for completeness is spotty, but this gives a clear target.