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Symmetric X_9 Singularities and Complex Affine Reflection Groups

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To Vladimir Igorevich
on the occasion of his 70th birthday

Abstract—We establish a natural correspondence between the finite order automorphisms of the function singularities X_9 and the complex crystallographic groups. A complete list of the related objects is obtained.

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Relations between singularities and Coxeter groups is a classical area of singularity theory going back to the fundamental works by Arnold [1] and Brieskorn [6]. Recently it was observed that these relations can be extended to include symmetric simple function singularities on the one hand and certain Shephard–Todd groups on the other [11–13]. In this paper we are making a further natural step in this direction by relating symmetries of the function singularities X_9 to a number of Popov's complex crystallographic groups [16]. Appearance of complex affine reflection groups in equivariant monodromy of parabolic function singularities with symmetry is the first appearance of such groups in any singularity context (see also [14]).

The structure of the paper is as follows.

Section 1 introduces the crystallographic groups which will be related to the function singularities. In addition, in Subsection 1.2 we describe a way to construct a complex affine reflection group from a semi-definite Hermitian form of corank 1.

Section 2 lists finite order automorphisms of the X_9 functions. It also shows how the rank 2 kernel of the X_9 Hermitian intersection form is shared by various character subspaces H_χ of the symmetry action on the middle vanishing homology.

Section 3 is devoted to the proof of the main result of the paper that all the complex affine reflection groups arising from the equivariant monodromy of the symmetric X_9 singularities on the appropriate H_χ via the construction of Subsection 1.2 are actually crystallographic.

1. AFFINE REFLECTION GROUPS

1.1. The complex crystallographic groups. An *affine reflection* in $\mathbb{C}^n$ is an affine unitary transformation identical on a hyperplane. The hyperplane is called the *mirror* of the reflection. A group generated by such reflections and having a compact fundamental domain is called *complex crystallographic*. These groups were classified by V.L. Popov in [16].

For a complex crystallographic group W, we denote by $L \subset U_n$ its linear part, that is, the image of W under the natural map $W \to U_n$. The group L is a Shephard–Todd group. Let T be the maximal translation subgroup of W. Then W is an extension of L by T. Unlike the real case, W

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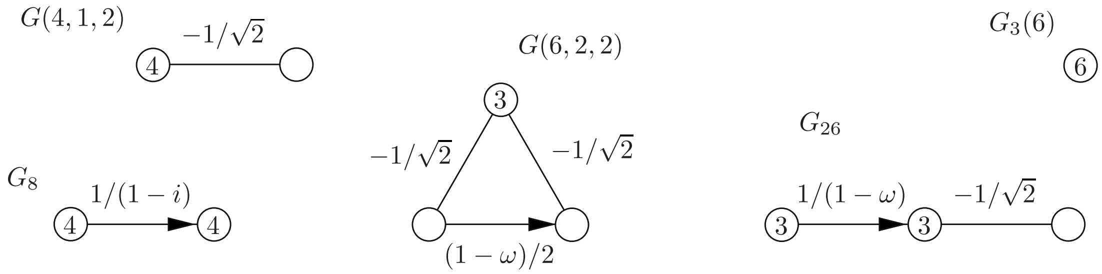

Fig. 1. Dynkin diagrams of the Shephard–Todd groups. All roots are unit.

may not be a semi-direct product of its linear and translation parts. However, all the groups we will need in our current singularity context are such products.

We will now describe the five groups to be involved. Mirrors of L will be identified by their normals, which we will call *roots*.

The linear parts of the groups we will need are the Shephard–Todd groups $L = G(4,1,2)$, $G(6,2,2), G_3(6), G_8, G_{26}$ (see [17, 16]). Their Dynkin diagrams are given in Fig. 1. The vertex set of a diagram represents a set of generating reflections. Each vertex is a unit root and is marked with the order of the reflection, order 2 omitted. An edge $a \to b$ is equipped with the Hermitian product $\langle a, b\rangle$. As usual, $\omega = e^{2\pi i/3}$. The edge orientation is omitted if the product is real, and there is no edge at all if the roots are orthogonal. All the diagrams were constructed using the roots from Table 2 of [16] (see also [9]). The rank of the group $G(6,2,2)$ is 2. The rank of any other group is equal to the number of vertices in its diagram.

In the notation of [16], the crystallographic groups W with the above linear parts that will be related to function singularities in this paper are $[G(4,1,2)]_2$, $[G(6,2,2)]_2$, $[K_3(6)]$, $[K_8]$ and $[K_{26}]_1$. The lattice T is spanned by the L-orbit of any root of L of order 2 in the first two cases, of any root in the next two, and of any root of order 3 in the last case.

All the crystallographic groups have the conjugate versions, with i and ω replaced by their conjugates. However, the conjugations yield the same groups.

1.2. Affine groups defined by corank 1 Hermitian forms. The relation between the crystallographic groups and function singularities that we are going to establish is based on the following construction of a complex reflection group from a corank 1 Hermitian form (cf. [5]).

Let $\widetilde{q}$ be a corank 1 semi-definite Hermitian form on $\widetilde{V} = \mathbb{C}^{n+1}$. Choose a basis $e_0, e_1, \dots, e_n$ in $\widetilde{V}$ so that e_0 is in the kernel K of the form. The span of the $e_{j>0}$ will be denoted by V, and v will stand for the V-component of $\widetilde{v} \in \widetilde{V}$: $\widetilde{v} = v_0 e_0 + v$. In all the matrix expressions below, with a slight abuse of notation, elements $v \in V$ will be treated as columns of their coordinates $v_{j>0}$. For example, $\widetilde{q}(\widetilde{v}, \widetilde{w}) = v^T Q \overline{w}$, where $Q = (\widetilde{q}(e_i, e_j))_{i,j>0}$ is the matrix of the restriction $q = \widetilde{q}|_V$.

We consider the space $\widetilde{V}^*$ dual to $\widetilde{V}$ as $K^* \oplus V$. For coordinates on it we choose $\alpha_0, \alpha_1, \dots, \alpha_n$, so that a linear functional $\widetilde{\alpha}$ on $\widetilde{V}$ is written as

$$\widetilde{\alpha}(\widetilde{v}) = v_0\alpha_0 + v^T Q\alpha = v_0\alpha_0 + q(v, \overline{\alpha}).$$

Take a *pseudo-reflection* on $\widetilde{V}$ (that is, a transformation given by the same formula as a reflection if the form $\widetilde{q}$ were non-degenerate) with a root $\widetilde{u} \notin K$ and the eigenvalue λ:

$$A\colon \widetilde{v} \mapsto \widetilde{v} - (1-\lambda)\widetilde{q}(\widetilde{v}, \widetilde{u})\widetilde{u}/\widetilde{q}(\widetilde{u}, \widetilde{u}) = \big(v_0 + \gamma q(v,u)u_0\big)e_0 + \big(v + \gamma q(v,u)u\big),$$

where $\gamma = (\lambda - 1)/q(u,u)$. For the dual transformation A^*, we have

$$(A^*\widetilde{\alpha})\widetilde{v} = \widetilde{\alpha}(A^{-1}\widetilde{v}) = v_0\alpha_0 + v^T Q\alpha + \overline{\gamma} v^T Q(u_0\alpha_0 + u^T Q\alpha)\overline{u} = v_0\alpha_0 + v^T Q(\alpha + \overline{\gamma}\widetilde{\alpha}(\widetilde{u})\overline{u}).$$

Therefore, the dual transformation sends each of the hyperplanes $\alpha_0 = \text{const}$ into itself, and on such a hyperplane it acts as

$$\alpha \mapsto \alpha + \overline{\gamma}\widetilde{\alpha}(\widetilde{u})\overline{u} = \alpha - (1-\overline{\lambda})\frac{\alpha_0 u_0 + \overline{q}(\alpha, \overline{u})}{\overline{q}(\overline{u}, \overline{u})}\overline{u}, \tag{1}$$

where $\overline{q}$ is the Hermitian form on V conjugate to q: it has the matrix $\overline{Q} = Q^T$ in the basis $e_{j>0}$. If $\alpha_0 \neq 0$, then this is an affine reflection on the hyperplane $\Gamma = \{\alpha_0 = \text{const}\} \simeq V$, with the root $\overline{u}$, mirror $\widetilde{\alpha}(\widetilde{u}) = \alpha_0 u_0 + \overline{q}(\alpha, \overline{u}) = 0$ and eigenvalue $\overline{\lambda}$. For $u_0 = 0$, the transformation is linear.

2. SMOOTHABLE SYMMETRIES OF X_9

Now we introduce the function singularities we will be dealing with.

Let f be a holomorphic function-germ on $(\mathbb{C}^n, 0)$, with an isolated singularity at the origin. Consider a diffeomorphism-germ g of $(\mathbb{C}^n, 0)$ sending the hypersurface $f = 0$ into itself. It multiplies f by a function c not vanishing at the origin. In what follows we assume g is of a finite order, so c is a constant, a root of unity.

Let $\mathcal{O}(g, c)$ be the space of all holomorphic function-germs on $(\mathbb{C}^n, 0)$ multiplied by c under the action of g. The group $\mathcal{R}_g$ of biholomorphism-germs of $(\mathbb{C}^n, 0)$ commuting with g acts on $\mathcal{O}(g, c)$. The corresponding equivalence is a geometric equivalence in the sense of Damon [10]. Therefore, the base of an $\mathcal{R}_g$-miniversal deformation of f in $\mathcal{O}(g, c)$ is smooth and such a deformation can be constructed in the standard way [10, 4].

Definition 2.1. An automorphism g of a hypersurface $f = 0$ is called *smoothable* if an $\mathcal{R}_g$-versal deformation of the function f contains members with smooth zero sets.

If g is such an automorphism, then the zero level M of a generic member of an $\mathcal{R}_g$-versal deformation is a g-invariant Milnor fibre of f. Hence, g acts on the homology of M and provides the splitting

$$H_{n-1}(M, \mathbb{C}) = \bigoplus\nolimits_\chi H_\chi, \qquad \chi^{\mathrm{order}(g)} = 1, \tag{2}$$

of the middle homology, in which g acts on an individual summand as a multiplication by the character χ. The *equivariant monodromy group*, that is, the monodromy within an $\mathcal{R}_g$-versal deformation of f, preserves the splitting. The monodromy action on the H_χ will be our source of complex crystallographic groups, upon the application of the construction of Subsection 1.2.

We restrict our attention to the classification of smoothable automorphisms of curves of the X_9 family

$$x^4 + ax^2y^2 + y^4 = 0, \qquad a^2 \neq 4. \tag{3}$$

The classification is up to holomorphic changes of the coordinates. Since our actual major aim is to obtain the homology splitting (2), we will not distinguish between automorphisms generating the same cyclic groups. Moreover, we prefer to have a Hermitian intersection form on the middle homology rather than skew-Hermitian. Because of that, we stabilise equation (3) by adding z^2 to the left-hand side. Respectively, g starts acting on z by the multiplication by one of the two possible square roots of c. We call this action *stabilised*. The ambiguity in choosing a root affects only the character assignment in (2) but not the direct summands themselves. Since only the summands are crucial for us, we give just one of the choices in our classification. In particular, we set g act trivially on z if the function is g-invariant.

Theorem 2.1. *The complete list of stabilised smoothable automorphisms of all X_9 curves is given in the table.*

Symmetric X_9 singularities

f	$g\colon x,y,z\mapsto$	$\lvert g\rvert$	Versal monomials	Kernel χ	Affine group	Notation
$x^4+y^4+z^2$	$ix,-y,z$	4	$1,y^2,x^2y$	$\pm i$	$[G(4,1,2)]_2$	$X_9\vert\mathbb{Z}_4$
	ix,y,z	4	$1,y,y^2$	$\pm i$	$[K_8]$	$A_3^{(4)}$
	$\omega x,i\omega y,\overline{\omega}z$	12	x	$\pm i$	–	$X_9/\mathbb{Z}_{12}$
$x^4+xy^3+z^2$	$ix,i\omega y,z$	12	1	$-\omega,-\overline{\omega}$	–	$X_9\vert\mathbb{Z}_{12}$
	$-x,-\omega y,z$	6	$1,x^2$	$\omega,\overline{\omega}$	$[K_3(6)]$	$B_2^{(6,3)}$
	$x,\omega y,z$	3	$1,x,x^2,x^3$	$\omega,\overline{\omega}$	$[K_{26}]_1$	$C_4^{(2,3)}$
	$\omega x,\overline{\omega}y,-\overline{\omega}z$	6	x,y^2,x^2y	$-\omega,-\overline{\omega}$	$[G(6,2,2)]_2$	$X_9/\mathbb{Z}_6$
	$\varepsilon_9x,\varepsilon_9^4y,\varepsilon_9^2z$	9	y	$\omega,\overline{\omega}$	–	$X_9/\mathbb{Z}_9$
$x^3y+xy^3+z^2$	$\varepsilon_8x,-\varepsilon_8y,z$	8	1	$\pm i$	–	$X_9\vert\mathbb{Z}_8$
$x^4+ax^2y^2+y^4+z^2$	$ix,-iy,z$	4	$1,xy,x^2y^2$	1	–	$(X_9\vert\mathbb{Z}_4)'$
	$-x,y,z$	2	$1,y,y^2,x^2,x^2y,x^2y^2$	-1	–	$K_{4,2}$
	$\omega x,-\omega y,\overline{\omega}z$	6	x,x^2y^2	-1	–	$(X_9/\mathbb{Z}_6)'$
	ix,iy,z	4	$1,x^2y^2$	-1	–	$(X_9\vert\mathbb{Z}_4)''$
	$-x,-y,z$	2	$1,x^2,xy,y^2,x^2y^2$	1	–	$(X_9\vert\mathbb{Z}_2)''$
	$\omega x,\omega y,\overline{\omega}z$	3	x,y,x^2y^2	1	–	$X_9/\mathbb{Z}_3$

In the table

- $\varepsilon_r=e^{2\pi i/r}$;
- the *versal monomials* are those that we add with arbitrary coefficients to f to obtain an $\mathcal{R}_g$-miniversal deformation;
- the *kernel* χ are the values of the character for which the restrictions of the Hermitian intersection form from $H_2(M,\mathbb{C})$ to the H_χ are degenerate (see Proposition 2.1 below);
- the *affine groups* are the complex crystallographic groups that will be constructed in Section 3 from the monodromy on the H_χ on which the intersection form has corank 1;
- similar to [7, 11], if the discriminant of a symmetric function singularity coincides with that of a Weyl group, the group enters the *notation*, the superscripts indicating the orders of the Picard–Lefschetz operators (see Section 3);
- the $K_{4,2}$ is the unimodular boundary function singularity of [2, 3];
- in all the other cases, the notation shows the symmetry group of the singularity, with the vertical line telling that the function is invariant under the action and the slash indicating that it is equivariant (cf. [13, 12]).

About the proof of Theorem 2.1. The classification process is based on the consideration how the automorphism permutes the four branches of the curve (3). The smoothability is heavily restricted by an obvious observation that, once a smoothable diffeomorphism of the plane has been diagonalised, it multiplies the function f by the same factor by which it multiplies one of the monomials 1, x and y (otherwise the zero level of any symmetric perturbation of f would have a critical point at the origin). The rest of the classification is rather straightforward. □

For the application of the construction of Subsection 1.2, it is crucial to know how the rank 2 kernel of the X_9 Hermitian intersection form is shared by the character subspaces.

Proposition 2.1. *The kernel values of the character χ for the symmetric X_9 singularities are those given in the table.*

Proof. We distinguish between invariant and equivariant cases, that is, when 1 either is or is not among the versal monomials.

(a) In the invariant cases, the kernel characters are the eigenvalues of the action of g on the residue forms $dx\,dy\,dz/df$ and $q_4(x,y)dx\,dy\,dz/df$, where $q_4(x,y)$ is a degree 4 monomial defining a non-trivial element in the local algebra of f. The span of the two forms is dual to the kernel of the intersection form on the homology.

(b) We do the equivariant functions case-by-case, using mainly the fact that the cycles in the kernel of the intersection form are invariant under any monodromy.

$X_9/\mathbb{Z}_3$. The monodromy $\alpha = e^{2\pi it}$, $0 \le t \le 1$, in the family $f(x,y,z) + \alpha x = 0$ coincides with the transformation g; hence all the kernel of the X_9 intersection form is in $H_{\chi=1}$.

$X_9/\mathbb{Z}_9$. The top-dimensional strata of the discriminant of $X_9/\mathbb{Z}_3$ are $3A_1$ only. Three ordinary Morse 2-cycles e, ge and g^2e vanishing simultaneously provide an element

$$e + \chi^{-1}ge + \chi^{-2}g^2e \in H_\chi, \qquad \chi^3 = 1. \tag{4}$$

This implies that the ranks of all the three H_χ are the same, 3. On the other hand, the automorphism of $X_9/\mathbb{Z}_3$ is the cube of that of $X_9/\mathbb{Z}_9$. Hence the kernel characters of $X_9/\mathbb{Z}_9$ are cubic roots of unity. Since the kernel character set must be sent into itself by the complex conjugation, we see that for $X_9/\mathbb{Z}_9$ the kernel of the X_9 form is spanned by the one-dimensional spaces H_ω and $H_{\overline{\omega}}$.

$X_9/\mathbb{Z}_{12}$. Take $M = \{x^4 + y^4 + z^2 - x = 0\}$ as a symmetric Milnor fibre. It retracts to the $\mathbb{Z}_{12}$-orbit of the 2-cell $\sigma = \{(x,y,z)\colon 0 \le x \le 1,\ y \ge 0,\ z \in \mathbb{R}\} \subset M \cap \mathbb{R}^3$. The linear combination

$$\sum_{j=0}^{11} \chi^{-j} g^j \sigma, \qquad \chi^{12} = 1, \quad \chi^3 \ne 1, \tag{5}$$

spans H_χ. On the other hand, the quasi-homogeneous monodromy in the family

$$x^4 + y^4 + z^2 - e^{2\pi it}x = 0, \qquad 0 \le t \le 1,$$

is g^4. Hence the kernel characters satisfy $\chi^4 = 1$. With $\chi = 1$ prohibited, this gives $\chi = \pm i$.

$(X_9/\mathbb{Z}_6)'$. The square of the $X_9/\mathbb{Z}_{12}$ automorphism is the inverse of that of $(X_9/\mathbb{Z}_6)'$. So, the above implies that the kernel of the X_9 form is now the rank 2 space $H_{\chi=-1}$.

$X_9/\mathbb{Z}_6$. The deformation $f(x,y,z) + \alpha x^2 y$ gives an adjacency of $X_9/\mathbb{Z}_6$ to the singularity $D_6/\mathbb{Z}_6$ of [13, 12], all of whose H_χ, $\chi^3 = -1$, are of rank 2. The multiplicity of the $X_9/\mathbb{Z}_6$ discriminant is 4, one higher than that of $D_6/\mathbb{Z}_6$, the increase due to the $3A_1$ stratum. This implies that the dimension of each of the three character spaces of $X_9/\mathbb{Z}_6$ is 3. Since the ranks of the intersection forms on them are at least 2, the characters $-\omega$ and $-\overline{\omega}$ are kernel. $\square$

Questions 2.1. (a) A bit more careful calculations show that, for all symmetric X_9 singularities, the rank of a character subspace with a degenerate intersection form is equal to the dimension of the base of an equivariant miniversal deformation, that is, to the number of the versal monomials. The same is true for the J_{10} symmetries of [14]. Why is this so?

(b) It would be also good to understand why the kernel of the intersection form does not split exactly when the symmetric singularity has a module.

3. RELATING SYMMETRIC X_9 SINGULARITIES AND CRYSTALLOGRAPHIC GROUPS

We call a symmetric X_9 singularity *interesting* if the monodromy group on one of its character subspaces gives rise to an affine complex reflection group (not necessarily crystallographic) via the construction of Subsection 1.2. Necessary conditions for this are as follows:

- the rank 2 kernel of the X_9 Hermitian intersection form splits between two character subspaces;
- each of the two subspaces must be of rank at least 2;
- the multiplicity of the discriminant of a symmetric singularity must be at least 2, since an affine reflection group has at least two generators which must be coming from the Picard–Lefschetz operators.

According to the table, the first condition eliminates all moduli cases. The last condition eliminates four further singularities with one-dimensional bases of miniversal deformations. This leaves exactly five interesting symmetries, those to which the table assigns affine groups.

In Fig. 2 the discriminants of three interesting X_9 singularities are shown. The degeneration types to which the top strata correspond are indicated. The $X_9/\mathbb{Z}_6$ discriminant is that of B_3 with an additional smooth component. The ordering α, β, γ of the deformation parameters is by the increase of their quasi-homogeneous weight in the deformations using the versal monomials of the table. The equation of the $X_9|\mathbb{Z}_4$ discriminant is

$$\gamma(\beta^2 - 4\gamma)\bigl((\beta - \alpha^2/4)^2 - 4\gamma\bigr) = 0.$$

Two discriminants missing from Fig. 2 are those of the singularities $A_3^{(4)}$ and $C_4^{(2,3)}$. The first of them is the standard A_3 swallowtail, with the top stratum A_3. The second is the standard C_4 discriminant with the smooth and singular components $3A_1$ and A_2, respectively.

The main result of this paper is

Theorem 3.1. *Consider an interesting symmetric X_9 singularity. Let χ be one of its kernel characters and Γ the hyperplane in H_χ^* formed by all linear functionals taking a fixed non-zero value on a fixed element of the kernel of the Hermitian intersection form on H_χ. Then the equivariant monodromy group of the singularity acting on Γ is the complex crystallographic group given in the table.*

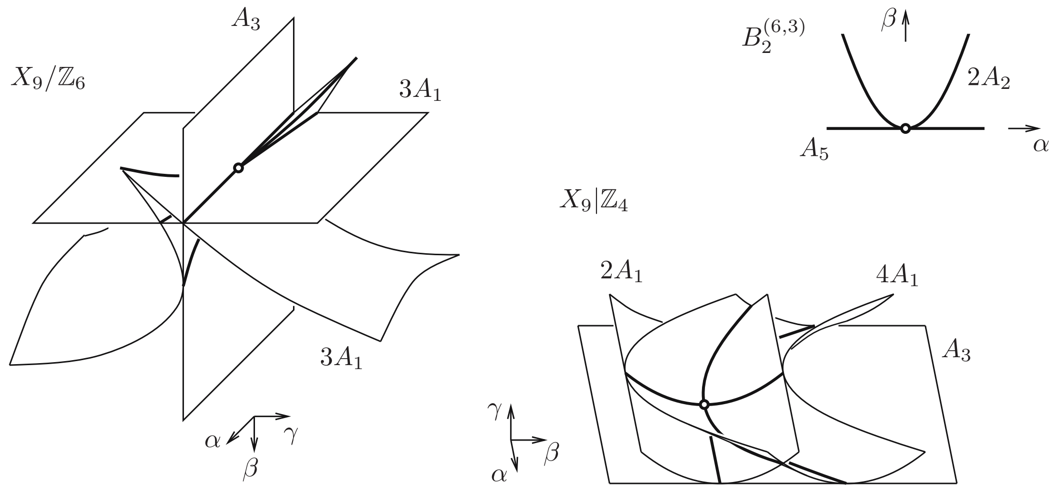

Fig. 2. Discriminants of the symmetric X_9 singularities.

Proof. By the methods developed in [11–13], it is possible to construct, for each H_χ of the theorem, distinguished sets of vanishing χ-cycles whose Dynkin diagrams are those of Fig. 3. The sets are bases of the H_χ, except for the $X_9/\mathbb{Z}_6$ case which has one relation.

We use the following conventions in the diagrams. The vertices are elements of a distinguished set of χ-cycles. A χ-cycle vanishing at a kA_ν stratum has the self-intersection number $-k(\nu+1)$, which is written at the vertex. The order of the corresponding Picard–Lefschetz operator is $\nu+1$ (written inside the vertex, order 2 omitted). Simple, double and triple edges indicate that the relations between the pairs of the operators are $aba = bab$, $(ab)^2 = (ba)^2$ and $(ab)^3 = (ba)^3$, respectively. The marking and orientation of the edges are similar to those in Fig. 1.

The idea behind the cycle construction is as follows. Consider the quotient set $M' = M/\mathbb{Z}_m$ of a symmetric Milnor fibre by the group generated by the automorphism g. This set is stratified according to the stationary subgroups of the points. Let $M'' \subset M'$ be the union of all strata whose dimension is less than $\dim M'$. When the deformation parameter approaches its discriminantal value, it is easy to define geometrically a relative vanishing cycle in (M', M''). Let $\sigma \subset M'$ be this cycle and $\sigma_0, \sigma_1, \dots, \sigma_{m-1}$ its inverse images in M ordered so that $g(\sigma_j) = \sigma_{(j+1) \bmod m}$. Then $\sum_{j=0}^{m-1} \chi^{-j}\sigma_j \in H_\chi$ is the χ-cycle we are looking for. The cycles (4) and (5) are examples of the construction.

Each tree diagram of Fig. 3 serves both kernel values of the character since vanishing χ-cycles are defined up to multiplication by powers of χ and change of orientation.

A vanishing χ-cycle defines the Picard–Lefschetz operator on H_χ. This is a pseudo-reflection with the cycle as its root. Thus we are ready to apply the construction of Subsection 1.2. To introduce the notations used in it, we denote by $e_0', e_1, e_2, \dots$ the vertices in each tree diagram going from left to right, and in the $X_9/\mathbb{Z}_6$ diagram starting from the top left and going clockwise (in this case $e_3 = -\overline{\chi}e_1 + \chi e_2$). For all the singularities, a generator of the kernel of the Hermitian intersection form can be taken in the form $e_0 = e_0' + \mathbf{a}$, where $\mathbf{a}$ is a linear combination of the $e_{j>0}$. The vector $\mathbf{a}$ will be called the *truncated kernel vector*. It is an analog of the negative of the maximal root of a Weyl group.

Now drop the vertex e_0' from each tree diagram of Fig. 3, change the sign of the intersection form and divide all the roots by appropriate positive numbers to make all of them unit. The result will be exactly the diagrams of Fig. 1 of the linear parts L of the affine groups assigned to the singularities by the table (for $\chi = -\overline{\omega}$ in the $X_9/\mathbb{Z}_6$ case, the additional complex conjugation is required). Therefore, the Picard–Lefschetz operators corresponding to the χ-cycles $e_{j>0}$ define the Shephard–Todd group L on the hyperplane $\Gamma \subset H_\chi^*$.

The translation vector of the transformation (1) is proportional to its root, as it should be in an affine reflection. Thus, Theorem 3.1 will be proven if it turns out that, in all the cases, the truncated kernel vector $\mathbf{a}$ is a root of a reflection from L of the order specified at the end of Subsection 1.1.

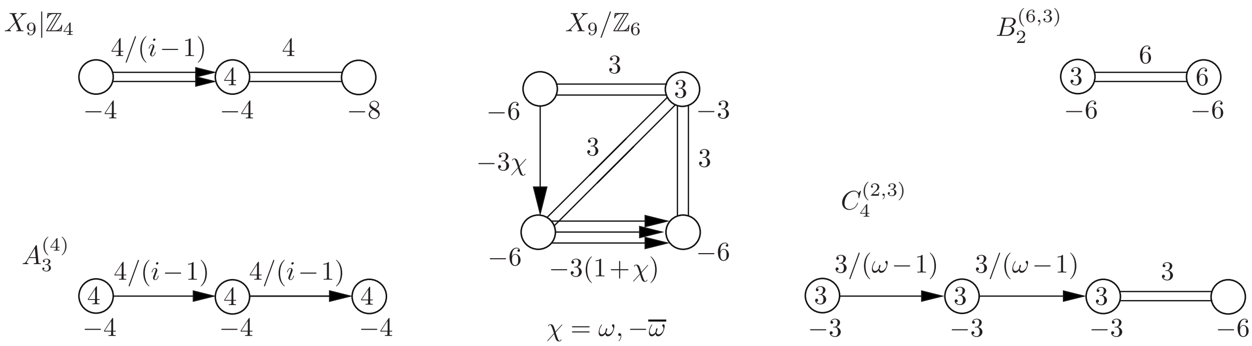

Fig. 3. Dynkin diagrams of the symmetric X_9 singularities in three variables.

And, indeed, we have

$$
\begin{aligned}
X_9|\mathbb{Z}_4\colon &\quad \mathbf{a} \sim (2,1) = A_1^2 e_2,\\
A_3^{(4)}\colon &\quad \mathbf{a} = (-1-i, i) = -A_1 A_2^{-1} e_1,\\
B_2^{(6,3)}\colon &\quad \mathbf{a} = e_1,\\
C_4^{(2,3)}\colon &\quad \mathbf{a} \sim (\omega - 1, 2, 1) = -A_1^{-1} A_2 A_3 A_2 e_1,\\
X_9/\mathbb{Z}_6\colon &\quad \mathbf{a} = -2e_1 - e_2 = \begin{cases} A_1 e_3, & \chi = -\omega,\\ A_1^{-1} e_3, & \chi = -\overline{\omega}, \end{cases}
\end{aligned}
$$

where the vector $\mathbf{a}$ or its multiple are written in the basis $e_{j>0}$ and the A_j are the linear reflections defined by the roots e_j and having the eigenvalues $-1, \omega, i$. This yields the result required. $\square$

Remarks 3.1. (a) The eigenvalue of the Picard–Lefschetz operator corresponding to a multiple Morse degeneration is -1. The eigenvalues of all the other operators in the $X_9|\mathbb{Z}_4$ and $X_9/\mathbb{Z}_6$ singularities are $-\chi$. They are χ in the $A_3^{(4)}$ and $C_4^{(2,3)}$ cases. Finally, for the $B_2^{(6,3)}$ singularity, the operators of orders 3 and 6 have the eigenvalues χ and $-\overline{\chi}$, respectively. This follows from easy quasi-homogeneous considerations similar to those in [11–13].

(b) The standard order of vanishing cycles in the distinguished set used to construct the $X_9/\mathbb{Z}_6$ diagram is e_2, e_0', e_1, e_3. As usual, for a tree diagram the order may be made arbitrary.

(c) The three crystallographic groups corresponding to the three symmetric X_9 singularities with the Weyl groups in the notations are representations of the corresponding generalised braid groups.

We should also notice that the fact that the equivariant monodromies of Theorem 3.1 are at most factor groups of the crystallographic groups in question already follows from the description of the discriminants of our singularities and the information about the orders of the Picard–Lefschetz operators. Indeed, consider first the four string diagrams of Fig. 3 omitting their edge orientations and all the labellings. Applying Zariski's method to calculate the fundamental groups of the complements to our discriminants, we see that the reduced diagrams are exactly the diagrams of the relations between the generators of these groups. If we now restore the orders of the vertices, then we come to the diagrammatic presentations of the corresponding crystallographic groups obtained in [15]. To obtain similar coincidence with [15] for $X_9/\mathbb{Z}_6$, we use the interpretation of the triple intersection of the discriminant: the lower right triangle of the Dynkin diagram corresponds to the

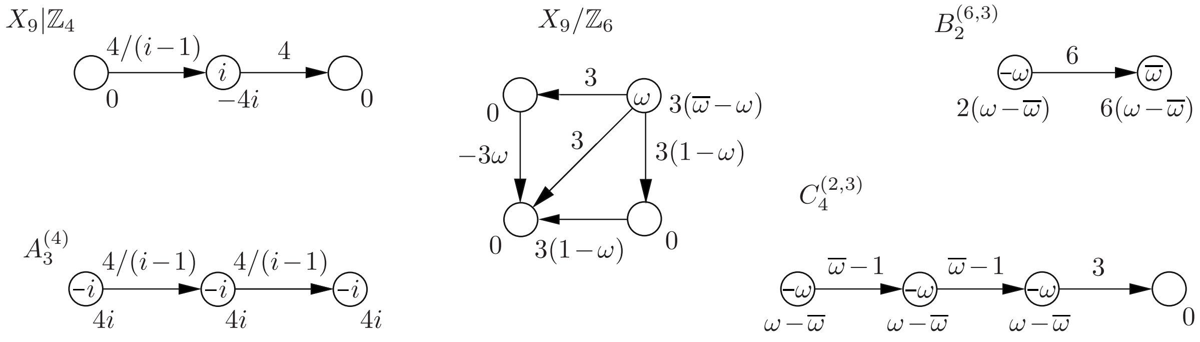

Fig. 4. Dynkin diagrams of the symmetric X_9 singularities in two variables, $\chi = i, \omega$.

circular relations $abc = bca = cab$ in the fundamental group (see [8, 15]). Finally, the additional relations in [15] are the orders of the classical monodromy in our cases.

Question 3.1. The relation between the discriminant of an interesting symmetric parabolic function and the orbit space of the related crystallographic group should be investigated. In particular, it would be interesting to find out why function singularities with non-isomorphic discriminants may give rise to the same crystallographic groups. At the moment, there are two examples of such a duplication: symmetric J_{10} singularities with the discriminants G_2 and C_3 (see [14]) correspond to the same affine groups, $[K_3(6)]$ and $[K_8]$, as respectively the singularities $B_2^{(6,3)}$ and $A_3^{(4)}$ of this paper.

The skew-Hermitian versions of the five affine reflection groups are given by the Dynkin diagrams of the two-variable symmetric X_9 singularities of Fig. 4. The diagrams are drawn for $\chi = i$ and $\chi = \omega$ for the two-variable automorphisms of the table of orders 4 and 3 or 6, respectively. For $\chi = -i, \overline{\omega}$, all the numbers must be conjugated. Inside the vertices are the eigenvalues of the Picard–Lefschetz operators. The empty vertices correspond to the kA_1 degenerations; hence all the eigenvalues for them are 1 and the Picard–Lefschetz operators are $a \mapsto a - \langle a, e \rangle e/k$. The three cycles forming the lower right triangle of the $X_9/\mathbb{Z}_6$ diagram are linearly dependent.

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