Structure / Nonstructure in Finite Model Theory

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

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in fact, my friend, it's not safe to make thin cuts; it's safer to go along cutting through the middle of things, and that way one will be more likely to encounter real classes. ... whenever there is a class of anything, it is necesarily also a part of whatever it is called a class of, but it is not at all necessary that a part is a class.

Plato, *Statesman* (C. J. Rowe trans.),
lines 262b and 263b in Stephanus.

Decision Problems with finite constraints

Tameness problems for classes of finite structures

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- $\mathbb{P}$: a property of subsets of $\mathcal{Q}$
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Two cases of interest:

- $\mathbb{P} = \mathit{WQO}$: No infinite antichain
- *Universality:* The class has a universal countable limit.

I WQO

- *Worst case scenario*
- *A Finiteness Theorem*
- *Concrete Cases*

II Universality

- *Beyond the pale*
- *Within the pale*

1 WQO

2 Universality

WQO

Theorem (Harvey Friedman)

There is a computable well-founded partial ordering $\leq^*$ of $\mathbb{N}$ for which the set

$$\{n \in \mathbb{N} : (\mathbb{N}, \leq^*)_n \text{ is WQO}\}$$

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$\leq^* = \leq \cap \leq_1$ — Then $(\mathbb{N}, \leq^*)_n$ is WQO iff $(\mathbb{N}, \leq_1)_n$ is WO

A Finiteness Theorem

Theorem (CL2000)

Let $\mathcal{Q}$ be a well-founded quasiorder and k fixed. Then there is a finite set Λ_k of infinite antichains such that:

$$\forall C \subseteq \mathcal{Q} \text{If } C \subseteq \mathcal{Q},\ |C| \leq k, \text{ and } \mathcal{Q}_C \text{ is not WQO,}$$
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Construction:

$$\Lambda_{k+1} = \Lambda_k \cup \bigcup_{I_1,\ldots,I_{k+1} \in \Lambda_k} \{I_C : \text{critical } C \text{ in } \prod_i \mathcal{Q}^{<<I_i}\}$$

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Question: Does this happen in the cases of interest?
(Yes in one simple case: vertex colored paths)

Cases of Interest: Graphs

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- Graphs with forbidden induced subgraphs:

Unclear ...

Cases of Interest: Tournaments

- Tournaments:

$\Lambda_1 = \{I_1, I_2\}$: $I_1 = \{N_{1,n,D} : n \geq 7\}$, $I_2 = \{N_{2,2n+1,H} : n \geq 4\}$.

$N_{k,n}$: Linear of order n, but with successors and edges (i, j) with $i \equiv j \mod k$ reversed.

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A set of finite tournaments determined by one "forbidden tournament" is wqo iff the infinite antichains I_1, I_2 are incompatible with the constraint.

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Corollary

The WQO problem for classes of tournaments determined by at most two forbidden tournaments is (p-time) decidable.

Remark. No actual bound on the degree of the polynomial

. . .

Cases of Interest: Pattern Classes

- Pattern Classes of Permutations

Theorem (Knuth, 1969)

The permutations which can be sorted using a stack are those omitting the pattern (231) *and their number is given by the Catalan numbers (cf. Macmahon 1915).*

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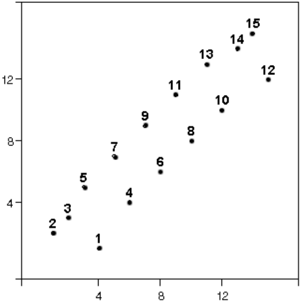
Themes:

- Characterize permutations sortable by variations on stacks
- Algorithmic problems for such classes of permutations
- Rates of growth for the the numbers of such permutations
- WQO (Antichains)

Tournaments: Infinite Antichains

An antichain:



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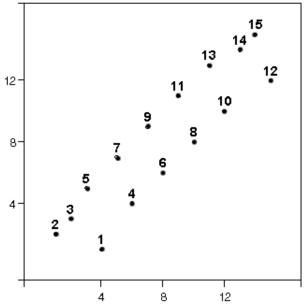

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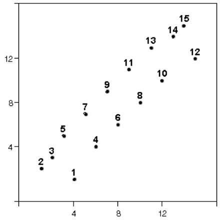

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References:

N. Ruškuc, "Decidability questions for pattern avoidance classes of permutations," in *Third International Conference on Permutation Patterns, Gainesville, Fla., 2005*

S. Waton, *On Permutation Classes Defined by Token Passing Networks, Gridding Matrices, and Pictures: Three Flavours of Involvement*, Ph.D. Thesis, St. Andrews, 2007

The Main Question

Once more:

$\mathcal{Q}$: Finite relational structures with signature σ (with symmetry conditions).

- Isolated points are dense?
- Isolated points are effectively given ($\mathcal{Q}^{<<I}$ decidable)?

1. WQO
2. Universality

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$(\mathcal{Q}, \leq)$: weak substructure.
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Examples (Graphs)

- Forbid K_n (Henson via Fraïssé)
- C a set of cycles: *forbid odd cycles up to some fixed size* (CSS,1999)

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...why the difference? ...

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To forbid a pattern places a condition on induced subgraphs.

Algebraic Closure

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Structural Analysis

Algebraic Closure

$\mathrm{acl}_{\mathcal{C}}(A)$: e.g., if you bound the vertex degree by forbidding a star, then the algebraic closure of a point is its connected component.

Theorem

If the algebraic closure operator is locally finite then the model completion of the theory of $\mathcal{C}$-free graphs is $\aleph_0$-categorical (and its model is universal).

Other tameness conditions

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- Number of models

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- Number of models
- Stability and its kin (for the associated theory).

Question

Is stability (and so on) a decidable property, as a function of the constraint set $\mathcal{C}$? Is this combinatorially interesting (or robust) at the finite level?