Metrically homogeneous graphs and distance semigroups

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Themes

- Metrically homogeneous graphs
- Classification problem
- KPT theory (combinatorics/topological dynamics) and . . .
- The distance semigroups $D^{\delta}_{M,C}$ (the Czech New Wave[1]).

[1]". . . a common sense of humour, absurdity, pathos, and sometimes startling surrealism."—Wikipedia

Definition

A connected graph is metrically homogeneous if it is homogeneous as a metric space, under the shortest path metric.

A metric space is homogeneous in the sense of Urysohn if Euclidean and Kleinian congruence agree on finite configurations.

Urysohn space

Rational Urysohn space: $\mathbb{U}_{\mathbb{Q}}$ (universal homogeneous $\mathbb{Q}$-space)

Urysohn space: $\mathbb{U}$ (its completion)
A universal complete separable metric space.

The Urysohn graph: $\mathbb{U}_{\mathbb{Z}}$ with the distance 1 edge relation.
A metrically homogeneous graph.

A [quite] strong homogeneity condition

…eine recht starke Homogenitätsbedingung …man den ganzen Raum (isometrisch) so auf sich selbst abbilden kann, dass dabei eine beliebige endliche Menge M in eine ebenfalls beliebige, der Menge *M* kongruente Menge *M*1 übergeführt wird.

Alexandrov to Hausdorff, Batz-sur-Mer, Aug. 3, 1924, reprinted in Hausdorff's collected works (cited by Hušek)

3° U *est homogène* en ce sens que, les ensembles finis et congruents A et B (situés dans U) étant quelconques, il existe une représentation isométrique de U sur lui-même transformant A en B.

CRAS, March 16, 1925 (posthumous)

The two Pavels

Alexandroff und Urysohn

From Hausdorff's collected works, vol. IX: correspondence.

References

Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity, Loren Graham and Jean-Michel Kantor, Harvard UP

Cf. `https://harvardpress.typepad.com/hup_publicity/2012/03/the-end-of-the-luzin-affair.html` (2012).

Pavel Alexandrov, *Pages from an Autobiography*, Russian Mathematical Surveys **34** (1979), 267–302 and **35** (1980) 315–358, esp. 318–319.

Classification problem

Problem

Classify the homogeneous complete geodesic spaces.

Example (Tent-Ziegler 2013, unpublished)

There is a countable homogeneous geodesic S-metric space with $S \subseteq \mathbb{R}$ dense, whose completion is not homogeneous.

The discrete case

Proposition (Cameron, 1998, given more generally)

Metrically homogeneous graphs are the distance 1 graphs of integral geodesic homogeneous metric spaces.

Continuous/Discrete

Continuous	*Discrete*
Homogeneous geodesic complete metric spaces	Metrically homogeneous graphs
2-point homogeneous	Distance transitive
Birkhoff, Busemann 1941–44	

Compact	*Loc. compact*	*Finite*	*Loc. finite*
H.-C. Wang 1951	Tits 1952, Freudenthal 1956	ongoing	Macpherson 1982

Birkhoff, 1944

METRIC FOUNDATIONS OF GEOMETRY. I

BY

GARRETT BIRKHOFF

1. **Introduction.** It is shown below by elementary methods that n-dimensional Euclidean, spherical, and hyperbolic geometry can be characterized by the following postulates.

Postulate I. Space is metric (in the sense of Fréchet).

Postulate II. Through any two points a line segment can be drawn, *locally* uniquely.

Postulate III. Any isometry between subsets of space can be extended to a self-isometry of all space.

⋮

Postulate III is not to be confused with the frequently stated weaker condition: "If M and N are isometric sets, then there exists a self-isometry of space which carries M into N." It should also be distinguished from the *n-point homogeneity* condition: "Any isometry between two sets of n or fewer points can be extended to a self-isometry of space." Thus Hilbert space has n-point homogeneity for every finite order n, yet does not satisfy the free mobility postulate; the same is true of Urysohn space (P. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. vol. 51 (1927) pp. 43–64, 74–90).

Freudenthal, 1956

Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems *)

ISSAI SCHUR zum Gedächtnis

Von

HANS FREUDENTHAL

- Riemann 1854 (1867); Helmholtz 1868; Lie (1870–)1890
- Hilbert 1902 (appendix IV)
- Brouwer, Moore, Süß, Cairns, Lubben, Kérekjartó (1909–1950)
- Montgomery and Zippin (1940)
- Kolmogorov 1930; Birkhoff 1941, 1944; Busemann 1941,1942; Tits 1952

The (discrete) Classification Problem

We also do not know whether every distance homogeneous graph is distance finite. In the countable case, an answer to this question might be a step towards a classification of the distance homogeneous graphs. We should mention in this

(Moss, 1992)

By contrast, the theory of infinite distance-transitive graphs is open. Not even the countable metrically homogeneous graphs have been determined. The purpose of

(Cameron, 1998)

Raising the stakes: 2005

FRAÏSSÉ LIMITS, RAMSEY THEORY, AND TOPOLOGICAL DYNAMICS OF AUTOMORPHISM GROUPS

A.S. KECHRIS, V.G. PESTOV AND S. TODORCEVIC

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

European Journal of Combinatorics 28 (2007) 457–468

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Metric spaces are Ramsey

Jaroslav Nešetřil

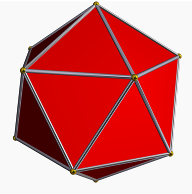
(KPT, 2005; Nešetřil, 2005 (pub. 2007))

Exploration: A catalog (2011)

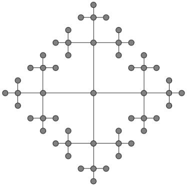
The Catalog

Non-generic type		*Generic type*
Finite	*Infinite*	6-parameter family
Classified (Cameron)	Classified (Macpherson, Cherlin)	

Examples: Non-Generic Type

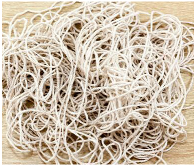


Icosahedron: Finite, antipodal, diameter 3, $\Gamma_1 = C_5$.

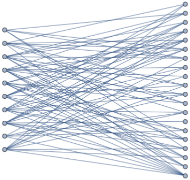


4-Regular Tree: Locally finite, Infinite diameter, Γ_1 edgeless

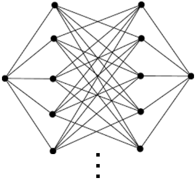
Examples: Generic Type



Urysohn graph
of diameter δ: Γ^δ.



Generic bipartite graph
of diameter δ.



Generic antipodal
bipartite graph
of diameter 3.

Definitions

Definition (Generic type:)

- Γ_1 is a primitive infinite graph.
- The common neighbors of two vertices at distance 2 contain an infinite independent set.

Definition (3-constrained)

A homogeneous structure is *3-constrained* if its minimal forbidden substructures are of order at most 3.

E.g.: the Urysohn graph.

A classification theorem

Theorem

The minimal constraints for 3-constrained metrically homogeneous graphs form a 5-parameter family uniformly definable in Presburger arithmetic.

$$\mathcal{T}(\delta; K_1, K_2, C_0, C_1) \text{ (forbidden triangles)}$$

Example

(a_1) Constraint C_0: no triangle has even perimeter $p \geq C_0$.
(a_2) Constraint K_1; no triangle of odd perimeter below $2K_1 + 1$.
(b) If $\min(C_0, C_1) \leq 2\delta + K_1$ then $C_1 = 2K_1 + 2K_2 + 1 \leq C_0$.

Notation

$$\Gamma^{\delta}_{K_1, K_2, C_0, C_1}$$

The sixth element

Definition

A *Henson constraint* for $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ is a $(1,\delta)$-space if Γ is not antipodal; otherwise, it is a $(1,\delta-1)$-space.

Conjecture (Catalog)

The metrically homogeneous graphs of generic type are defined by constraint sets of the form

$$\mathcal{T}(\delta; K_1, K_2, C_0, C_1) \cup \mathcal{S}$$

where $\mathcal{S}$ is a set of Henson constraints.

Notation

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1;\mathcal{S}}$$

The main theorem

Theorem

Let Γ be a known primitive metrically homogeneous graph of generic type and finite diameter δ.
Let $\mathcal{A}^w$ be the class of $[\delta]$-edge labeled graphs embedding as weak substructures of Γ. Then $\mathcal{A}^w$ is finitely determined.

In other words, there are finitely many minimal forbidden weak substructures up to isomorphism.

Example

For the Urysohn graph of diameter δ, the minimal forbidden weak substructures are the *non-metric cycles* consisting of an edge of one length and a path of shorter total length.

References

Special cases, with applications, in Rebecca Coulson's thesis.

More or less general case: Andrés Aranda, David Bradley-Williams, Eng Keat Hng, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and Micheal Pawliuk, *Completing graphs to metric spaces*, Electronic Notes in Discrete Mathematics **61** (2017), 53–60

More briefly: The Prague cabal.
Revisited in Konečný's BA and MA theses (with semigroups in the MA version).
One may alsoo compare Hubička's habilitation for a broad view.

More on the foundations in preparation, in theory—though other things tend to take precedence. An account by Hubička, Konečný, Nešetřil is hoped for as well.

Proof

1. Define a canonical completion procedure
2. Analyze obstructions.

Example (Urysohn graph)

Completion: Shortest path metric.
Obstructions: Non-metric cycles.

Remark

The completion procedure gives an amalgamation procedure if the class is a strong amalgamation class.
It also gives a notion of direct sum assuming the disjoint joint embedding property

Neutral distances

Notation

$$A \oplus_M B$$

Definition

M is a neutral distance for Γ if the subspaces of Γ are closed under M-direct sums.

In the absence of perimeter bounds one generally takes $M = \delta$.

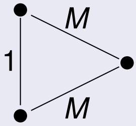
The neutral distances

Lemma

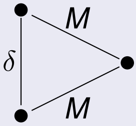
For the known metrically homogeneous graphs of generic type, the neutral distances are characterized by

$$\max(K_1, \delta/2) \leq M \leq K_2, (\min(C_0, C_1) - \delta - 1)/2$$

Proof (necessity).



$K_1 \leq M \leq K_2$



$\delta/2 \leq M < (C_\epsilon - \delta)/2$

$\square$

Existence

Lemma

For a known metrically homogeneous graph of generic type, without Henson constraints, a neutral distance exists if and only if the graph is primitive or antipodal of even diameter.

Bipartite: $K_1 = \infty$ (too big).
Antipodal: $C = 2\delta + 1$ (dangerously small: $M \leq \delta/2$)

For Henson constraints exclude $M = \delta$ (if not already excluded).

$+_{M,C}$

Definition

$x +_{M,C} y$ is the best approximation to M in the interval
$$[|x - y|, \min(x + y, C - (x + y) - 1)]$$

Lemma (Prague)

If Γ is a known primitive metrically homogeneous graph of generic type with $\min(C_0, C_1) = C$ and neutral distance M then $+_{M,C}$ is an associative operation on $[\delta]$.

$D^\delta_{M,C}$

Definition

$D^\delta_{M,C}$ is the partially ordered semigroup on $[\delta]$ with operation $+_{M,C}$ and partial order $\leq_{M,C}$ typically given by the natural order $\leq_{\text{nat}}$:

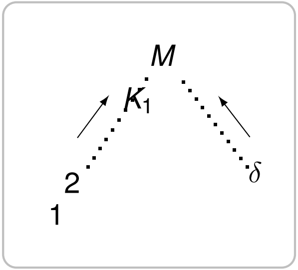
$$x \leq x +_{M,C} y$$

If $M = K_1$ a slightly different order may be needed.

Compare Sauer: $S \subseteq \mathbb{R}$ finite. $x +_S y = \max_S(z :\leq x + y)$, $\leq$ inherited from $\mathbb{R}$.

The class of S-spaces has the amalgamation property iff $+_S$ is associative (and the shortest path metric gives a canonical amalgam).

$D^{\delta}_{M,C}$



$D_{M,C}$
$(M + x = M)$

"Cross-relations" not shown. E.g. $1 \leq_{M,C} \delta - 1$.

Note—the "shortest path length" is defined as an infimum, so a priori may not lie in the set of path lengths. Here though the "shortest path length" is actually the length of the shortest path.

Shortest path completion

Theorem (Prague)

Primitive metrically homogeneous graphs of known type with associated distance semigroup $D^{\delta}_{M,C}$ are generalized metric spaces:

$$d(x,y) \leq_{M,C} d(x,z) +_{M,C} d(z,y)$$

and shortest path completion is a completion procedure.

For the proofs

The following conditions are jointly equivalent to amalgamation holding. Most but not all are needed for the completion result (so apparently the completion result holds in some cases where amalgamation does not).

I Bipartite $K_1 = \infty$:

- $K_2 = 0$, $C_1 = 2\delta + 1$; this is the *bipartite case*.

II Low $C_1 \leq 2\delta + K_1$ (K_1 finite)

- $\delta \geq 3$;
- $C_1 = 2K_1 + 2K_2 + 1 < C_0$;
- $K_1 + K_2 \geq \delta$;
- $K_1 + 2K_2 < 2\delta$;
- If $C_0 > C_1 + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$.

III High $C_0, C_1 \geq 2\delta + K_1 + 1$ (K_1 finite)

- $K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C_0, C_1 \geq 2\delta + K_1 + 2$;
- If $max(C_0, C_1) > \min(C_0, C_1) + 1$ then $C_0, C_1 \geq 2\delta + K_2$.

What is a distance semigroup?

Definition (Provisional)

A distance semigroup is a partially ordered semigroup $(D, +, \leq)$ for which $\leq$ extends $\leq_{\mathrm{nat}}$ and for which the class of D-metric spaces has amalgamation via shortest path completion.

Alternative: require a completion process. Or does that follow ? And is there an axiomatic definition?

Proposition (Braunfeld)

A lattice with its usual ordering is a distance semigroup if and only if it is distributive.

General distributive law:

$$x + \min(S) = \min(x + S)$$

when needed. (Whenever $\min(S)$ is defined?)

Classification: proof strategy

1. $\Gamma \mapsto (\delta, K_1, K_2, C_0, C_1, \mathcal{S})$.
2. Γ embeds into Γ^*, where
3. $\Gamma^* = \Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ exists.
4. Γ^* embeds into Γ (Embedding Theorem).

The proof of the embedding theorem is inductive (induction on diameter d of $A \subseteq \Gamma^*$) and starts with

$$d_0 = \max(K_1, 2)$$

Why?—Because one has good amalgamation properties above d_0 (clear if $d_0 = M_0 = \max(K_1, \lceil \delta/2 \rceil)$ by semigroup theory, and follows by iterating).

In progress, with Amato.
Induction step visible in ACM20—diameter 3 (see intro thereto).

Procrustean abstract

Moss and Cameron asked for the classification of the metrically homogeneous graphs; I conjectured an answer in 2011. It now appears that the classification conjecture can be proved by direct but lengthy methods (in progress, with Daniela Amato).

A cabal of combinatorialists found a useful way of looking at the known graphs as *generalized metric spaces* with values in a partially ordered semigroup (cf. Braunfeld, Conant, Sauer), using shortest path metrics in *partial* generalized metric spaces, generalizing Nešetřil, 2007.

The definition of the term "generalized metric space with values in a partially ordered semigroup" is not yet settled and the relevant framework does not coincide with any of those in the existing algebraic theory.

I thank Hubička and Konečný for multiple conversations, past and future.