Splitting Twisted Automorphism Groups

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1 Twisted Automorphisms

2 Random Edge Colorings

3 Twists of metrically homogeneous graphs

4 Lifting involutions

Structures and permutation groups

Structure		*Permutation Group*
A	$\longrightarrow$	$\mathrm{Aut}(A)$
A^k/G	$\longleftarrow$	G

Remark

A is homogeneous in the canonical language. (Orbits are isomorphism types.)

Twisted isomorphism

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Example (Homogeneous graphs)

Up to graph complementation, the infinite primitive homogeneous graphs are

- The random graph $\mathcal{R}$
- The generic K_n-free graphs $\mathcal{H}_n$, $n < \infty$ (Henson).

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$$\mathcal{H}_n \simeq^* \mathcal{H}_n^c$$

$$\mathcal{R} \simeq^* \mathcal{R} \text{ (with non-trivial twist)}$$

Twisted Automorphisms

The twisted automorphism group $\mathrm{Aut}^*(\Gamma)$.

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Example

$$\mathrm{Out}(\mathcal{H}_n) = 1 \qquad \mathrm{Out}(\mathcal{R}) = \mathrm{Sym}(2)$$

The splitting problem

Question (Cameron, Tarzi 2007)

When does the twisted automorphism group split?

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Example

The twisted automorphism group of $\mathcal{R}$ does not split.

(No involutory anti-automorphism of $\mathcal{R}$.)

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When does this split?

n	1	2	...
Splits?	✓	✗	

Table: Data

Splitting m-random graphs

Theorem (CT07)

For fixed n the following are equivalent.

- $\mathrm{Aut}^*(\mathcal{R}_n)$ *splits.*
- *n is odd.*

Splitting m-random graphs

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- $\mathrm{Aut}^*(\mathcal{R}_n)$ *splits.*
- *Every involutory twist lifts to an involutory twisted automorphism.*
- *Every involution in* $\mathrm{Sym}(n)$ *fixes a point.*
- *n is odd.*

Splitting *m*-random *p*-hypergraphs

Theorem

For fixed n and p prime the following are equivalent.

- $\mathrm{Aut}^*(\mathcal{R}_n^{(p)})$ *splits.*
- *Every twist of order p lifts to a twisted automorphism of order p.*
- *Every element of order p in* $\mathrm{Sym}(n)$ *fixes a point.*
- *n is not divisible by p.*

Splitting *m*-random *k*-hypergraphs

Theorem

For fixed n and k the following are equivalent.

- $\mathrm{Aut}^*(\mathcal{R}_n^{(k)})$ *splits.*
- *Every subgroup of* $\mathrm{Sym}(n)$ *whose order divides k fixes a point.*
- *(Every cyclic subgroup of* $\mathrm{Sym}(n)$ *whose order divides k fixes a point.)*
- *(n is not a sum of non-trivial divisors of k.)*

Example: $k = 6$, $n = 5$, $\tau = (12)(345)$.

Splitting m-random k-hypergraphs

Theorem

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Example: $k = 6$, $n = 5$, $\tau = (12)(345)$. Necessity.
$H \leq \mathrm{Sym}(n)$, order divides k.
Then H leaves some k-set invariant. □

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Target: A lifting of $\mathrm{Sym}(n)$ on $[n]$ to an action on $\mathcal{R}_n^{(k)}$.

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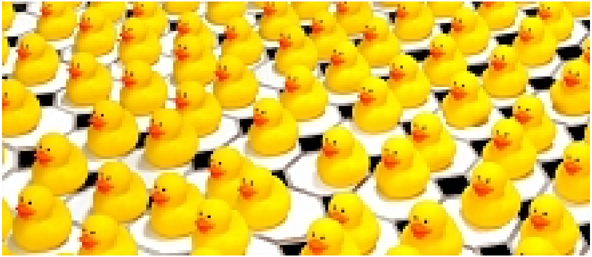

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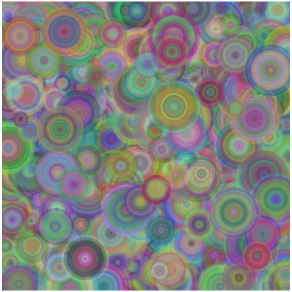

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  - Representative e, setwise stabilizer H: $H|$, divides $|e| = k$.
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Extension property

infinitely many chances with trivial setwise stabilizer, randomness wins
(new point in A_i, i very large).

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Metrically homogeneous graphs

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Examples

- Any connected homogeneous graph ($\delta = 2$).
- Random bipartite graph ($\delta = 3$).
- Regular tree $\delta = \infty$
- Urysohn graph, bounded Urysohn graph (any δ).

Some classification theorems

- Diameter 2: Lachlan/Woodrow 1980
- Diameter 3: Amato/Cherlin/Macpherson, preprint (78 pp.)
- non-generic type: finite or tree-like (Cameron,Macpherson,Ch2011)

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Definition (Generic type)

- The neighbors of a vertex form a random graph, a Henson graph, or an infinite independent set; and
- Definitely not a tree (girth at most 4 . . .)

Metric Twists

Theorem (Bannai/Bannai, Gardiner, 1980)

If a finite distance transitive graph of diameter δ and degree at least 3 is distance transitive with respect to two edge relations, then the distances are permuted according to one of four permutations ρ, ρ^{-1}, τ_0, τ_1 of the set $[\delta]$.

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Definition

- ρ: double the small distances up to $\delta/2$ and then work your way back down.
- τ_ϵ: interchange i with its “reflection” $(\delta + \epsilon - i)$ for $i \leq \delta/2$ odd.

The twists τ_ϵ have order 2.

Metric Twists, revisited

Theorem (Rebecca Coulson, 2019)

If there is a twisted isomorphism with a non-trivial twist between two metrically homogeneous graphs of diameter δ and generic type, then δ is finite and the distances are permuted according to one of the four permutations ρ, ρ^{-1}, τ_0, τ_1.

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If there is a twisted isomorphism with a non-trivial twist between two metrically homogeneous graphs of diameter δ and generic type, then δ is finite and the distances are permuted according to one of the four permutations ρ, ρ^{-1}, τ_0, τ_1.

Problem

Does this generalize to arbitrary distance transitive graphs?

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Theorem

If the outer automorphism group of a metrically homogeneous graph of generic type is non-trivial it is generated by τ_0 *or* τ_1.

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Problem

When does τ_ϵ lift?

Self-dual metrically homogeneous graphs

Theorem

The τ_0-self-dual metrically homogeneous graphs given by constraints on triangles are the following.

- *Generic bipartite antipodal*
- *Generic nearly bipartite antipodal (odd cycles appear first at length $2\lfloor\delta/2\rfloor+1$).*

With few exceptions, the τ_1-self-dual metrically homogeneous graphs given by constraints on triangles have a somewhat similar description.

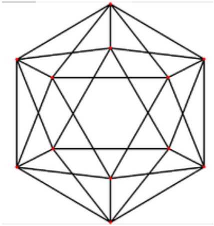
Self-dual metrically homogeneous graphs

Theorem

The τ_0-self-dual metrically homogeneous graphs given by constraints on triangles are the following.

- *Generic bipartite antipodal*
- *Generic nearly bipartite antipodal (odd cycles appear first at length $2\lfloor\delta/2\rfloor+1$).*

Antipodal: involutory symmetry at maximal distance.



Splitting Theorem

Theorem

Let Γ be one of the known metrically homogeneous graphs of generic type which is τ-self-dual ($\tau = \tau_\epsilon$). Then τ lifts to an involutory twisted automorphism of Γ if and only if the midpoint or midpoints of the reflected interval in $[\delta]$ are fixed.

Concretely:

$$\delta + \epsilon \not\equiv 3 \pmod 4$$

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Check amalgamation.

Question

Can one use the Prague “magic semigroup” to simplify the calculations?

Why fixed points in the middle?

α involutory twisted automorphism with twist τ

Suppose midpoints K_1, K_2 are interchanged ($\delta + \epsilon \equiv 3 \pmod 4$).
Look at the values

$$D = \{d(x, x,^{\alpha}) \mid x \neq x^{\alpha}\}$$
$$= D_0 \cup D_1 \text{ (small/large values, with a gap)}$$

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$$\begin{aligned} D &= \{d(x, x,^{\alpha})\,|\, x \neq x^{\alpha}\} \\ &= D_0 \cup D_1 \text{ (small/large values, with a gap)} \end{aligned}$$

- D_0, D_1 are both non-empty.
- $\exists\, x_1, x_2$ at distance 2 with $d(x_i, x_i^{\alpha}) = K_i + (-1)^i$
- $\exists\, a$ at distance 1 from x_1, x_2 and at distance K_1, K_2 from x_1^{α}, x_2^{α} respectively.

$$d(a^{\alpha}, x_1) = K_2 \qquad d(a^{\alpha}, x_2) = K_1$$

$$d(a, a^{\alpha}) \in [K_2 - 1, K_1 + 1] \text{ (contradiction)}$$

Λ-ultrametric spaces

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If Λ is a finite distributive lattice then there is a generic *generalized ultrametric space* U_Λ with values in Λ.

The outer automorphism group of U_Λ is

$$\mathrm{Aut}(\Lambda)$$

This can be any finite group.
When does this split?

Summing up

1. What are the relations in general between the following conditions in the binary homogeneous case (or replace 2 by any prime)?
   - Splitting the twisted automorphism group.
   - Lifting involutions.
   - Structure of fixed point sets for involutions.
2. Is there a general theory connecting fixed points of twists and some notion of generic automorphism?