Finite Homogeneous Structures
Dedicated to the memory of Mati Rubin
Revised 5/2/2018

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

April 23, 14:30

- Homogeneity
- Homogeneous graphs
- Lachlan's Finiteness Theorem
- Finite Structures with few 4-types
- The Relational Complexity of a Finite Structure

Anecdotes

I felt blessed by the opportunity to reconnect with Mati in Fall 2013 at the Hausdorff Institute, in the context of a program which paid considerable attention to issues relating homogeneity and geometry. While it was important to acknowledge his contributions to mathematics on this occasion, the impact of his personal qualities and the love of his colleagues was also much in evidence at the conference. The combination of rigorously high standards with a realistic appreciation of human frailty does not always come easily.

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Anecdotes

In my talk, I made a few remarks about formative moments in the lives of young mathematicians which for the most part I will omit here—but this was sparked by a remark Alex Lubotzky made to me in January 2018 about the impact of a summer camp for aspiring young mathematicians on his own relation to mathematics: he found that it opened him up to a sense of mathematics as a wonderfully welcoming community. This had something to do with his fellow summer campers and much to do with his camp counselor—none other than Mati Rubin. Alex attests that at least some of the qualities for which Mati became so widely appreciated and admired were already in evidence at that time.

The subject

It is very well known that Mati did major work on automorphism groups of homogeneous structures and homeomorphism groups of similar topological structures, many with a strongly geometrical character. My talk concerns on the one hand some historical points I have been trying to clarify, sparked in part by Hušek's article in the proceedings of the Beersheva conference on the Urysohn space edited by Arkady Leiderman, Mati Rubin, and others; and on the other hand, aspects of the theory of finite homogeneous structures which have involved substantial input from both model theorists and group theorists, and for which the ball is now largely in the group theorists' court.

Coming up:

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> *. . . a really strong condition of homogeneity: namely, the whole space may be mapped (isometrically) onto itself so as to carry over any finite set S into an equally arbitrary congruent set S_1. (In German; later, in French).*

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This concerns the last piece of work Urysohn carried out before his untimely death in a swimming accident, and it was left to his close friend and contemporary Pavel Alexandrov to prepare the work for publication.
In the volume of Hausdorff's correspondence included in his connected works, his last letter to Alexandrov was written on the anniversary of Urysohn's death, when Alexandrov had made a kind of pilgrimage back to the town where his friend died.

Geometrical Homogeneity

There is also a geometrical theory of *2-point homogeneity (distance transitivity):* see Birkhoff, Busemann, Tits, Wang, Szabó; and there is a discrete variant of this (distance transitive graphs).
The geometrical theory focuses on the locally compact case; this is analogous to the locally finite case in the context of graphs. More precisely, the compact case corresponds to the finite case (which is not yet completely treated) while the locally compact but not compact case corresponds to the infinite locally finite case (treated by Macpherson).

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Precursors: Cantor 1893 … Hausdorff 1914 $(\mathbb{Q}, <)$; Skolem 1920 $(\mathbb{Q}, <)$ with dense colors; Fraïssé 1953, Erdös–Rényi 1963, Rado 1964

But they had other things in mind.

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• The case of linear orders is perhaps even more striking.

Coming up:

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Relational Complexity of Finite Structures

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$$\left(\exists x_1 \cdots \exists x_n \left[\bigwedge_{i<j} x_i \neq x_j\right], n<\omega\right).$$

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See Brian Rotman or Ward Henson, 1971. (Rado was on Rotman's thesis committee.)

MR0308009 (46 #7124) 08A05
Rotman, B. **[Rotman, Brian]**
Remarks on some theorems of Rado on universal graphs.
J. London Math. Soc. (2) **4** (1971), 123–126.

Let i_0 and j_0 be ordinals and $n(i)$ and $m(j)$ finite ordinals, for $i < i_0$ and $j < j_0$. A structure of type (i_0, j_0, n, m) is an object $\mathbf{A} = (A, F_i, R_j)_i < i_0,\ j < j_0$, where A is a set, F_i an $n(i)$-ary function on A and R_j an $m(j)$-ary relation on A. The notion of embedding of one structure in another is defined in the obvious way. Let K be a class of structures. The structure $\mathbf{A}^*$ is said to be $(\aleph_\alpha, K)$-homogeneous if, for every $\mathbf{B} \in K$ with $B \subseteq A$ and $|B| < \aleph_\alpha$, every embedding of $\mathbf{B}$ in $\mathbf{A}^*$ can be extended to an automorphism of $\mathbf{A}^*$. The structure $\mathbf{A}^*$ is said to be universal in a class K of structures if every structure from K can be embedded in $\mathbf{A}^*$. B. Jónsson [Math. Scand. **4** (1956), 193–208; MR0096608; ibid. **8** (1960), 137–142; MR0125021], and M. Morley and R. Vaught [ibid. **11** (1962), 37–57; MR0150032] have established conditions for a given class K to possess a universal member that is homogeneous for a given $\aleph_\alpha$. The present author shows that most, though not all, of the results of the reviewer [Acta Arith. **9** (1964), 331–340; MR0172268] can be deduced as corollaries of the very general model-theoretic theorems described above.

R. Rado

Classification

Theorem (Lachlan/Woodrow 1980)

Up to complementation, the (countable) homogeneous graphs are:

- *The imprimitive (or degenerate) graphs* $m \cdot K_n$, $1 \leq m, n \leq \infty$.
- *The finite nondegenerate primitive graphs* C_5 *and* $K_3 \Box K_3 = AO_2^-(3)$.
- *The generic* K_n*-free graphs* Γ_n *(Henson 1973)*
- *The random graph* Γ_∞.

The Lachlan/Woodrow paper

Itay Kaplan pointed out to me that the 1980 paper expresses this result differently.
The abstract says (and elucidates) the following.

> THEOREM: *Let $\mathcal{G}_1, \mathcal{G}_2$ be two countable (infinite) ultrahomogeneous graphs such that for each $H \in \mathcal{D}$ H can be embedded in $\mathcal{G}_1$ just in case it can be embedded in $\mathcal{G}_2$. Then $\mathcal{G}_1 \simeq \mathcal{G}_2$.* COROLLARY: *There are a countable number of countable ultrahomogeneous (undirected) graphs.*

The meaning of this is made more transparent by putting the last three lines of the introduction to the paper together with the reduction of their Theorem 1 to Theorem 2 on page 53, and notably the brief use of Ramsey's theorem. Combined with the classification in the finite case by Gardiner or Sheehan, this produces the explcit form we gave.

Avatars

But what interests us now is the following.
(Lachlan): in Shelah's stable/unstable dichotomy, the stable class consists of two sporadic finite examples and the avatars of $\infty \cdot K_\infty$.

Definition (Avatar)

A smoothly embedded substructure of a homogeneous structure is one whose automorphism group has the same orbits on n-tuples as the full automorphism group.

Example

$m \cdot K_n \to \infty \cdot K_\infty$

Stable	*Unstable*
C_5, $K_3 \square K_3$	Γ_n, Γ_∞
$\infty \cdot K_\infty$ (family)	

Coming up:

The Theorem

Theorem (Lachlan)

Let L be a finite relational language. Then there are finitely many stable homogeneous L-structures $M_1, \ldots, M_n$ such that every stable homogeneous L-structure is a smoothly embedded substructure of the M_i. In particular, the finite homogeneous L-structures are the finite smoothly embedded substructures of the M_i.

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Remark

The theory as originally formulated uses Shelah's "complete rank" and applies to homogeneous L-structures with bounded complete rank, an analog of Morley rank which makes sense in finite structures (and has other virtues).

Bounding the Complete Rank

Lemma (Technical Lemma)

For a given finite relational language L, there is a bound on the complete ranks of homogeneous L-structures.

For binary languages there is a combinatorial approach [LS1984].

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In general, one also uses permutation group theory (CFSG).

Lemma (ChL1986, Lemma B)

Let n, s be fixed. Then for every sufficiently large permutation structure (X, G) satisfying

$$|X^5/G| \leq s$$

there is a set of indiscernibles of order n in X.

Coming up:

Finite relational languages

Definition

Relational complexity $\rho(M)$:
$\min(r : M$ is homogeneous in an r-ary language.)
$s_n(M)$: $|M^n/\operatorname{Aut}(M)|$.

Finite relational language: $\rho(M)$ finite and $s_\rho(M)$ finite.

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Finite relational language: $\rho(M)$ finite and $s_\rho(M)$ finite.
Lachlan's suggestion: drop the homogeneity but keep the bound on $s_n(M)$ with M finite.

Finiteness Theorem

Theorem (ChHr1990–2003, Thm 6 (2))

Let L be a finite language and k a natural number. Then the class of finite L-structures with $s_4 \leq k$ can be divided into families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ for some computable n such that each family is associated with a single countable Lie coordinatizable structure Γ_i and a formula ϕ_i such that the structures in $\mathcal{F}_i$ are finite homogeneous substructures of Γ_i, with isomorphism types characterized by invariants computable in polynomial time.

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Lemma (Kantor-Liebeck-Macpherson1989)

For fixed k, any sufficiently large finite primitive structure with $s_5 \leq k$ is known: coordinatized by a finite union of Lie or quadratic geometries of complexity $\max(e, |K|, \tau) \leq k$.

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Philosophy Lie geometries play the role of strongly minimal sets. E.g., vector spaces $V(q)$; possibly decorated with quadratic forms.

Breakdown of orthogonality theory:(V, V^*) has a *non-trivial interaction* between strongly minimal sets (arising from an outer automorphism of the automorphism group of V).

Neostability ....

Coming up:

Relational Complexity: The two problems

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Examples

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(1) $\rho(V) \approx \dim V$.
Fundamental relations: linear dependence
$y = \lambda(x_1, \ldots, x_n)$.
In other words, to understand V, do linear algebra.
(2) $\rho(P^1) = 4$: cross ratio
(3) $\rho([{n \atop k}]) = \lfloor \log_2 k \rfloor + 2$
Fundamental relations: sizes of atoms $\alpha(A_1, \ldots, A_r)$.
(Bounded as $n \to \infty$ with k fixed.)

The relational complexity of Alt_n on k-sets

Theorem

- *For the natural action of Alt_n on k-sets, $\rho = n - 3$, apart from the following exceptional cases.*

$$\begin{array}{ll} k = 1 & \rho = n - 1 \\ k = 2 & \rho = \max(n-2, 3) \\ k \geq 3,\ n = 2k+2 & \rho = n - 2 \end{array}$$

The relational complexity of Alt_n on k-sets

Theorem

... $\rho \geq n-3$, and usually equal.

$\rho([{n \atop k}], \mathrm{Sym}_n)$ is known to be small and $\rho([{n \atop k}], \mathrm{Alt}_n)$ is expected to be much larger.

So a "witness" for the value of ρ should be of the form $(X_1, \ldots, X_\rho)$ where the k-sets X_i separate points, together with the image of this sequence under an odd permutation—but on deletion of one X_i, there should be a pair (a_i, b_i) no longer separated.

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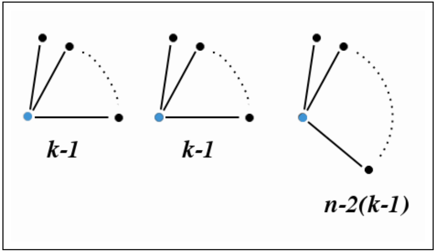
We make a graph Γ on $[n]$ with the ρ edges (a_i, b_i) separated by X_i, and only by X_i.
This is easily seen to be *acyclic,* so

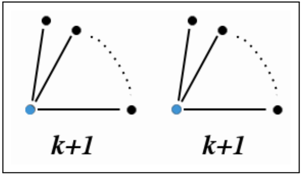
$$\rho = n - c$$

where c is the number of connected components. We claim the minimal value for c is 2 or 3 (for $k \geq 2$).

The graphs



$\rho \geq n - 3$



$n = 2k + 2$, $\rho = n - 2$

Some natural actions

- For the natural action of Sym_n on k-sets, $\rho = \lfloor \log_2 k \rfloor + 2$.

Problem

What is the relational complexity of the natural action of Sym_{nk} on partitions of shape $n \times k$?

Theorem (with Wiscons, 2017)

$\rho(n \times 2) = n$ *unless* $n = 1$ *or* 4*, in which case* $\rho = n + 1$*.*

Bounded ρ

Conjecture

Let Γ be primitive and binary. Then Γ is one of the following.

- *An indiscernible set.*
- *A cycle C_p, oriented or symmetric.*
- *An affine space over a finite field equipped with an anisotropic quadratic form.*

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Theorem (Wiscons2016)

If the conjecture fails, then it fails for some structure with an almost simple automorphism group: ($S \leq \mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(S)$).

$$\mathrm{Soc}(\mathrm{Aut}(\Gamma)) = S$$

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Theorem (To appear)

- *The conjecture holds when the automorphism group has socle an alternating group (Gill, Spiga) or a sporadic group (Dalla Volta, Gill, Spiga) or Lie rank 1 (Gill, Hunt, Spiga).*

Stay tuned . . . (Aschbacher classification)

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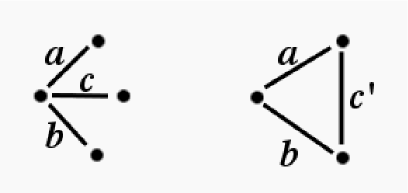

Insufficiently flexible.

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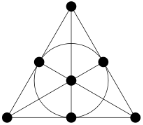
Fano plane:



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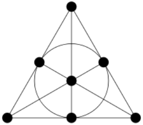
Fano plane:  With padding ($n \geq 9$), the induced action of Alt_n is doubly transitive on this set of triples.

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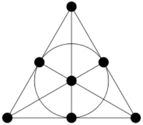
Fano plane:  With padding ($n \geq 9$), the induced action of Alt_n is doubly transitive on this set of triples.

Binarity would imply that the group induces Sym_7 on this set of triples, and that Alt_n induces at least Alt_7.

Surprisingly, this is a robust argument—even if the action is not known.

General actions with alternating socle

Key point: Usually, every element which fixes a point of X moves many points of n.

Babai, Liebeck, Saxl: at least $(\sqrt{n}-1)/2$.

Also: $\mathbb{F}_p^+ \rtimes \mathbb{F}_p^\times$ acting on $\mathbb{F}_p$ is a good source of doubly transitive actions.

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$0 \in X$, M the point stabilizer G_0.
If M does not act primitively then the action of G is known.
If M acts primitively, the key point applies.
One case requiring attention: if M again has simple socle.—

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Example

Suppose some element h of order 4 in M fixes a point of $[n]$.
Then we get an action of $\mathbb{F}_5^+ \rtimes \mathbb{F}_5^\times$ on both $[n]$ and X.

	$\mathbb{F}_5^+$	$\mathbb{F}_5^\times$
$[n]$	$g = (1, 2, 3, 5, 4)$	$h = (2, 3, 4, 5) \cdots$
X	$g = (0_X, 1_X, 2_X, 3_X, 4_X) \cdots$	$h = (1_X, 2_X, 4_X, 3_X) \cdots$

On X??—The 5-cycle g fixes many points, so is not in M.

What is wrong with this picture?

$(\mathbb{F}_5^+ \rtimes \mathbb{F}_5^\times, (0_X, 1_X, 2_X, 3_X, 4_X) \cdots)$ + Binarity ...

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$(\mathbb{F}_5^+ \rtimes \mathbb{F}_5^\times, (0_X, 1_X, 2_X, 3_X, 4_X)\cdots)$ + Binarity ...

We have a 5-cycle g (on $[n]$) inducing $(0, 1, 2, 3, 4)\cdots$ on X. If $t \in G$ induces $(1, 2)\cdots$ on X (binarity) then the element

$$[g, t]$$

acts as $(1, 2, 3)$ on X, so lies in $M = G_0$, while on $[n]$ it fixes all but 10 points, bounding n sharply.